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LINEAR LIGHT AMPLIFIER AND AMPLIFICATION OF N-PHOTON STATES

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Abstract

We consider a linear quantum amplifier consisting of N_A two-level atoms and study the problem of amplification of N-photon states. The N-photon states are associated with N-quantum states of the harmonic oscillator. We show that the process of interaction of the electromagnetic field with atoms can be associated with some transformation of the phase space and functions defined on this phase space. We consider the Husimi functions $Q_N(q,p)$ of N-quantum states of the harmonic oscillator, which are defined on the phase space, investigate transformation of these functions, and find an explicit form of the density matrix of the amplified N-photon state.

Keywords: quantum amplifier, Husimi function, harmonic oscillator, scaling transform, density matrix, phase space, evolution equation.

1. Introduction

We consider the process of interaction of a single-mode electromagnetic field with a two-level atomic medium. We assume that the medium is described by the Jaynes–Cummings model. We are interested in the dynamics of the electromagnetic field only; thus we assume that the state of the medium in the process of interaction does not change. In fact, the complete system is separated into two subsystems. Such separated systems are described by the density matrix [1–3]. The evolution equation satisfying the density matrix is known. It is solved by replacing the density operator with quasiprobability distributions.

The quasiprobability distribution functions such as Wigner functions [4], Glauber–Sudarshan functions [5,6], and Husimi–Kano functions [7,8] are often used in quantum optics. The Glauber–Sudarshan P-distribution was employed in [9–11] in order to obtain information on the properties of the field at the amplifier output. In [12], it was suggested to use the Husimi–Kano Q-distribution instead of the Glauber–Sudarshan P-distribution, the evolution equation for the Q-function was obtained, its solution was found for a particular case, and its properties were studied.

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In this paper, we use the original method, which was developed in [13–17] to analyze the solutions of the equation for the Husimi function. The essence of the method consists in the fact that a certain transformation of the phase space of the system is related to the dynamical process occurring in it. The transformations of the phase space generate the transformations of the functions defined on this phase space. Transformations of functions have different forms, depending on both the nature of the dynamical process and the type of functions defined on the phase space. In this paper, we consider the Husimi Q-functions. In [17], we have shown that in the case of the complete inverse population of two-level atomic systems forming the amplifier, the transformation of the phase space, corresponding to the amplification of the field passing through the amplifier reads

$$(q,p) \to (\lambda q, \lambda p), \qquad |\lambda|^2 \le 1. \tag{1}$$

It corresponds to scaling of the phase space, and the corresponding transformation of functions is

$$Q(q,p) \to \widetilde{Q}(q,p) = \lambda^2 Q(\lambda q, \lambda p). \tag{2}$$

In [13], we showed that if $Q(q,p)$ is the Husimi function of quantum state, then $\lambda^2 Q(\lambda q, \lambda p)$ is also the Husimi function of another quantum state, provided that $|\lambda|^2 \le 1$.

Here, we consider a more general situation, namely, we assume that a part of atoms forming the amplifier are in the ground state, and the rest of them are in the excited state. In this case, the dynamics of the process of interaction can also be associated with transformation of the phase space, but this transformation and the corresponding transformation of functions have a more complex form than (1) and (2). We describe these transformations and find an explicit form of the density matrix of an amplified N-photon state.

We treat the states of the harmonic oscillator only, but they can be identified as photons, which is a standard practice in quantum optics. In this way, the analysis of the amplification process and its results are also applicable for photons.

2. Quantum Amplifier Structure

We consider a system of N_A two-level atoms, N_1 of which are excited, and N_0 atoms are in the ground state, with $N_0 < N_1$. These atoms are interacting with the one-mode quantum field, for which we assume that it is an eigenmode of a free field, and that its frequency is resonant with the atomic frequency. We assume also that the populations N_1 and N_0 are kept constant in time due to some pump and loss mechanism.

Let $\hat{\rho}$ be the density operator of the electromagnetic field. The master equation for $\hat{\rho}$ reads [1, 12]

$$\frac{\partial \hat{\rho}}{\partial t} = -\gamma N_1(\hat{a}\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger) - \gamma N_0(\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}). \tag{3}$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the electromagnetic field, N_1 and N_0 are the populations of the upper and lower levels of the two-level atom, and γ is the amplification coefficient.

Using the relation between the density matrix and the Husimi function, we can pass from the operator equation (3) to an ordinary differential equation for the Husimi function. Using this equation, the expression for the Husimi function for the state at the quantum amplifier output was obtained in [12]; it is

$$Q(\alpha,t)=\int d^2\beta Q(\beta)\frac{1}{\pi m}\exp\left[-\frac{|\alpha-\beta G|^2}{m}\right]. \tag{4}$$

Here

$$G(t)=\exp[2(N_1-N_0)\gamma t],\qquad m=\frac{N_0}{N_1-N_0}(G^2-1). \tag{5}$$

In [17], we considered the case $m=0$ corresponding to the situation where all atoms are in the excited state. In this case, the Husimi function of the state after the amplification is given by a simple expression,

$$Q_{\rm out}(\alpha,t)=\frac{1}{G^2}Q_{\rm in}(\alpha/G)=\left\langle\frac{\alpha}{G}|\hat\rho_{\rm in}|\frac{\alpha}{G}\right\rangle. \tag{6}$$

We found an explicit form of the density matrix of the amplified state in the case where the input state is a superposition of an arbitrary number of Fock states.

The simplest case takes place if the input state is the pure N-photon state. In this case, at the output of the amplifier one has the state described by the following density matrix:

$$\hat\rho_N^{\lambda}=\sum_{j=0}^{\infty}\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!N!}|N+j\rangle\langle N+j|, \tag{7}$$

where $\lambda=G^{-1}$. The density matrix (7) is diagonal. Its first N diagonal elements with labels $0,1,\dots,N-1$ are equal to zero, and the remaining diagonal elements are

$$F_{N+j}^{N}=\frac{(1-\lambda^2)^j(N+j)!}{j!N!}\lambda^{2N+2},\qquad j=0,1,\dots \tag{8}$$

The values (8) form a negative binomial distribution, and the elements of this distribution are given by [19]

$$f(k,r,p)=\begin{pmatrix} r+k-1\\ k\end{pmatrix}p^rq^k=\frac{(r+k-1)!}{(r-1)!(k)!}p^r(1-p)^k,\qquad p+q=1,\quad k=0,1,2,\dots \tag{9}$$

They are defined by two parameters r and p, and the value k is the element label in the distribution. In our case, $r=N+1$, $k=j$, and $p=\lambda^2$, i.e., we have

$$F_{N+j}^{N}=f(j,N+1,\lambda^2). \tag{10}$$

The density matrix (7) of the amplified N-photon state was found in [17]. In this paper, we consider the case where the population levels N_1 and N_0 ($N_1>N_0$) can take arbitrary values. As an input state, we take the N-photon state and find the density matrix of the corresponding amplified state. We use formula (7) in our subsequent calculations. In order to distinguish the density matrices of amplified states obtained by an amplifier with $N_0=0$ from the density matrices of amplified states obtained by an amplifier with arbitrary N_0 and N_1, we denote the former by $\hat\rho$, and the latter by $\widetilde{\hat\rho}$.

3. The Density Matrix of Amplified State

In order to find the Husimi function $\widetilde{Q}(\alpha)$ of the amplified state, obtained due to passing trough a linear quantum amplifier with arbitrary values of populations N_1 and N_0 ($N_0 < N_1$), one should calculate the integral (4). The function $Q(\beta)$ is the Husimi function of the input state. For the input state, we take the quantum state of the harmonic oscillator $|N\rangle$; its Husimi function has the following form:

$$Q_N(\beta) = \langle\beta|N\rangle\langle N|\beta\rangle = e^{-|\beta|^2}\frac{|\beta|^{2N}}{N!}. \tag{11}$$

Substituting the above Husimi function into the integral (4) one obtains

$$\widetilde{Q}_N^\lambda(\alpha) = \frac{1}{\pi m N!}\int d^2\beta\, e^{-|\beta|^2}\,|\beta|^{2N}\,\exp\left[-\frac{|\alpha-\beta G|^2}{m}\right], \tag{12}$$

which after integration becomes

$$\widetilde{Q}_N^\lambda(\alpha) = \frac{1}{N!(m+G^2)}\frac{m^N}{(m+G^2)^N}L_N\left(-\frac{|\alpha|^2G^2}{m(m+G^2)}\right)\exp\left(-\frac{|\alpha|^2}{m+G^2}\right). \tag{13}$$

Here $L_N(x)$ is the Laguerre polynomial

$$L_N(x) = N!\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{x^k}{k!}. \tag{14}$$

In [17], we developed a method that allows one to derive the density matrix from the Husimi function of the form similar to (13). Applying this method, we obtain the density matrix $\widetilde{\hat{\rho}_N}^\lambda$ of the amplified $|N\rangle$ state; it reads

$$\begin{aligned}\widetilde{\hat{\rho}_N}^\lambda \;&=\; \frac{m^N}{(m+G^2)^{N+1}}\sum_{k=0}^{\infty}\sum_{l=0}^{N}\binom{N}{N-l}\binom{k+N-l}{n-l}\\ &\times\Big(\frac{G^2}{m(m+G^2)}\Big)^{N-l}\Big(1-\frac{1}{m+G^2}\Big)^k|k+N-l\rangle\langle k+n-l|,\end{aligned} \tag{15}$$

where $\lambda^{-2} = m + G^2$.

When $m = 0$, the summation over l reduces to only one term ($l = 0$), so that the density matrix becomes

$$\begin{aligned}\widetilde{\hat{\rho}_N}^\lambda(m\to 0) \;&=\; \frac{1}{G^{2(N+1)}}\sum_{k=0}^{\infty}\binom{k+N}{N}\Big(1-\frac{1}{G^2}\Big)^k|k+N\rangle\langle k+N|\\ &=\; \frac{1}{G^{2(N+1)}}\sum_{k=0}^{\infty}\frac{(n+k)!}{N!k!}\Big(1-\frac{1}{G^2}\Big)^k|k+N\rangle\langle k+N|.\end{aligned} \tag{16}$$

Now we consider some special cases. Let the input state at the entrance of the amplifier be the vacuum state $|0\rangle$. The Husimi function of $|0\rangle$ has the form

$$Q_0(\alpha) = e^{-|\alpha|^2}. \tag{17}$$

Substituting this expression into (12), we obtain the expression for the Husimi function of the amplified state $|0\rangle_{\rm out}$,

$$\widetilde{Q}_0^\lambda(\alpha) = \frac{1}{m+G^2}\exp\left(-\frac{|\alpha|^2}{m+G^2}\right). \tag{18}$$

We see that expression (18) coincides with the expression for the Husimi function of the state $|0\rangle_{\rm out}$ obtained in [17] assuming $\lambda^{-2} = m + G^2$. Therefore,

$$\widetilde{\hat{\rho}_0}^{\lambda} = \hat{\rho}_0^\lambda. \tag{19}$$

Let now the input state be $|1\rangle$. In this case, the state $|1\rangle_{\rm out}$ appearing at the amplifier output has the following density matrix:

$$\widetilde{\hat{\rho}_1}^{\lambda} = \frac{m}{(m+G^2)^2}\left[\sum_{k=0}^{\infty}\left(\frac{(k+1)G^2}{m(m+G^2)}\right)\left(1-\frac{1}{m+G^2}\right)^k |k+1\rangle\langle k+1|\right] + \frac{m}{(m+G^2)^2}\left[\sum_{k=0}^{\infty}\left(1-\frac{1}{m+G^2}\right)^k |k\rangle\langle k|\right]. \tag{20}$$

Comparing expression (20) with the density matrix of the amplified $|N\rangle$ state (7), we see that

$$\widetilde{\hat{\rho}_1}^{\lambda} = \frac{m}{m+G^2}\hat{\rho}_0^\lambda + \frac{G^2}{m+G^2}\hat{\rho}_1^\lambda. \tag{21}$$

In the general case, one has

$$\begin{aligned}\widetilde{\hat{\rho}_N}^{\lambda} &= m^N \sum_{l=0}^{N}\frac{N!\lambda^{2N+2}}{l!(N-l)!}\left(\frac{G^2\lambda^2}{m}\right)^{N-l}\sum_{k=0}^{\infty}(1-\lambda^2)^k\frac{(N-l+k)!}{k!(N-l)!}|N-l+k\rangle\langle N-l+k| \\ &= m^N\lambda^{2N}\sum_{l=0}^{N}\frac{N!}{l!(N-l)!}\left(\frac{G^2}{m}\right)^{N-l}\hat{\rho}_{N-l}^\lambda. \end{aligned}\tag{22}$$

We see that the density matrix (22) of the state, which is obtained from the N-photon state $|N\rangle$, after passing through the amplifier with arbitrary values of the population levels N_1 and N_0, is the sum of the density matrices $\hat{\rho}_{N-l}^\lambda$ of $(N-l)$-photon states passed through the amplifier, whose population of the ground level is $N_0 = 0$, i.e., trough the completely excited medium. Properties of such states were studied in [17] – they have the form (7). These density matrices are diagonal, with either zeros or values from the negative binomial distribution on the main diagonal. So in the density matrix $\hat{\rho}_{N-l}^\lambda$, the first $(N-l-1)$ numbers on the diagonal are zeros followed by values F_{N+j}^N, $j = 0, 1, \ldots$ given in (16). The total density matrix (22) is the sum of such matrices multiplied by the coefficients that are elements of the binomial distribution $\left(1+G^2/m\right)^N$. Thus, the matrix elements of the density matrix (22) have the form of sums of the products of the elements of the binomial distribution and the elements of the negative binomial distribution.

Let us note that in the case of a completely excited medium, when the population of the ground level is $N_0 = 0$, the N-photon amplified state, $|N\rangle_{\rm out}$, contains only states with photon number equal to or greater than N, namely, $|N\rangle$, $|N+1\rangle$, $|N+2\rangle, \ldots$ However, if the population of the ground level $N_0 > 0$, then the output state contains every k-photon state, including the vacuum state ($k = 0$).

Now we find the form of the diagonal elements of matrix (22); for this, we present expression (22) as

$$\widetilde{\hat{\rho}_N}^{\lambda} = \sum_{p=0}^{N} S_p^N \rho_p^{\lambda}, \tag{23}$$

where the coefficients S_p^N are

$$S_p^N = m^N \lambda^{2N} \left(\frac{G^2}{m}\right)^p \frac{N!}{p!(N-p)!} = m^{(N-p)} \lambda^{2N} G^{2p} \binom{N}{p}. \tag{24}$$

The density operator $\hat{\rho}_p^{\lambda}$ given in Eq. (16) can be rewritten as follows:

$$\hat{\rho}_p^{\lambda} = \sum_{j=0}^{\infty} F_{p+j}^{p} |p+j\rangle\langle p+j|, \tag{25}$$

where F_{p+j}^{p} are elements of the negative binomial distribution.

Now we can write the density operator given by (22) as

$$\widetilde{\hat{\rho}_N}^{\lambda} = \sum_{k=0}^{\infty} H_k^N |k\rangle\langle k|, \tag{26}$$

Here, the coefficients H_k^N are

$$H_k^N = \sum_{j=0}^{k} S_j^N F_k^j, \quad k = 0, 1, \ldots, N, \qquad H_k^N = \sum_{j=0}^{N} S_j^N F_k^j, \quad k = N+1, N+2, \ldots \infty. \tag{27}$$

Let us evaluate the trace of the density operator (22)

$$\begin{aligned}
\mathrm{Tr}(\widetilde{\hat{\rho}}_N^{\lambda}) &= \mathrm{Tr}\left(m^N \lambda^{2N} \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} \left(\frac{G^2}{m}\right)^{N-l} \rho_{N-l}^{\lambda}\right) \\
&= m^N \lambda^{2N} \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} \left(\frac{G^2}{m}\right)^{N-l} \mathrm{Tr}\left(\sum_{j=0}^{\infty} F_{p+j}^{p} |p+j\rangle\langle p+j|,\right) \\
&= m^N \lambda^{2N} \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} \left(\frac{G^2}{m}\right)^{N-l} \sum_{j=0}^{\infty} F_{p+j}^{p} \\
&= \left(\frac{m}{m+G^2}\right)^N \sum_{l=0}^{N} \binom{N}{l} \left(\frac{G^2}{m}\right)^{N-l} = 1.
\end{aligned} \tag{28}$$

Thus, by direct calculation we showed that the trace of the operator (22) is equal to unity. Operator (22) is the diagonal operator and the sum of its diagonal elements $\sum_{k=0}^{\infty} H_k^N = 1$. The trace of the operator $(\widetilde{\hat{\rho}_N}^{\lambda})^2$ reads

$$\mathrm{Tr}\left[(\widetilde{\hat{\rho}_N}^{\lambda})^2\right] = \sum_{k=0}^{\infty} (H_k^N)^2 \leq 1. \tag{29}$$

Therefore, operator (22) is actually the density operator of the quantum state.

4. Conclusions

The aim of this paper was to investigate the relation between the dynamics of physical processes and the transformations of the phase space and the functions defined on the phase space. The problem was addressed by considering a linear quantum amplifier. In this case, we found explicitly interrelations between the elements describing the dynamics and the transformations in the phase space.

We showed that the state of the quantum amplifier has a significant influence on the state of the electromagnetic field which passed trough it. In the case where the quantum amplifier is completely inverted ($N_0 = 0$), the transformation of the phase space and Husimi functions defined on it are determined by formulas (1) and (2). If both levels of the quantum amplifier, ground and excited, have nonzero population, the transformation of Husimi functions has a more complex form. It is determined by formula (4) which, in the particular case where the input state is the ground state of harmonic oscillator, after calculating the integral, takes the form (18). Expression (18) is structurally similar to the Husimi function transformed by (2), but the fundamental difference between transformations (2) and (4) consists in the fact that (2) is a local transformation, and (4) is a nonlocal transformation. Thus, the general structure of transformations of the phase space and the functions specified on the phase space is much more complex than considered in [17].

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