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# T-dualization of a weakly curved background

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**Abstract.** We consider a string moving in a weakly curved background, composed of a constant metric and a linearly coordinate dependent Kalb-Ramond field with an infinitesimal strength. We discuss the T-dualization procedure which we developed for a closed bosonic string moving in a weakly curved background. The procedure is a generalization of a Buscher T-dualization procedure and enables the T-dualization of the nonisometry directions. The same procedure is used to investigate the T-duals of an open bosonic string as well. The generalized T-dualizations give insight to the connection between the geometrical properties of the T-dual spaces.

## 1. Introduction

In string theory there exists a symmetry, T-duality, which allows the physical equivalence of the string living on the different geometrical structures of the compactified dimensions. The string living in a space with one dimension compactified on a radius $R$, has the same physical features as a string leaving in a space with one dimension compactified on a radius $\frac{\alpha'}{R}$, where $\alpha'$ is a Regge slope parameter. T-duality was first described in the context of toroidal compactification in [1] (thoroughly explained in [2]), it can be generalized to the arbitrary toroidal compactification [3], and extended to the non-flat conformal backgrounds [4]. The origin of T-duality is seen in a possibility that, unlike a point particle, the string can wrap around compactified dimensions.

The first T-dualization procedure, the prescription for obtaining a theory which is T-dual of a given theory, was defined by Buscher [5]. The procedure was done for a string sigma model, describing a string moving in a background composed of a metric $G_{\mu\nu}$, an antisymmetric field $B_{\mu\nu}$ and a dilaton field $\Phi$. It is required that the metric admits at least one continuous abelian isometry which leaves the action invariant. The procedure is founded in gauging the isometry by introducing the gauge fields $v^\mu_\alpha$. In order to preserve the physical content of the original theory, one requires that the new fields $v^\mu_\alpha$ are nonphysical, which is achieved by the requirement that the gauge fields have a vanishing field strength $F^\mu_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha$. This requirement is included in the theory by adding the Lagrange multiplier term $y_\mu F^\mu_{01}$ into the Lagrangian. Fixing the gauge one obtains the gauge fixed Lagrangian which carries the information on both initial and a T-dual theory. The integration over the Lagrange multipliers $y_\mu$, simply recovers the original theory. The integration over the gauge fields $v^\mu_\alpha$, produces the $T$-dual theory.

The standard T-dualization procedure is applicable along directions which do not appear as the background field arguments. The generalized T-dualization procedure which is applicable along an arbitrary coordinate was done in Refs. [6, 7, 8]. The procedure is founded in the standard procedure and keeps the main rules of the standard procedure. In order to gauge the global isometry, one introduces the gauge fields $v^\mu_\alpha$, as usual. The replacement of the derivatives $\partial_\alpha x^\mu$ with the covariant ones $D_\alpha x^\mu$, does not as before make the whole action invariant. The

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obstacle is the background field $B_{\mu\nu}$ depending on $x^\mu$, which is not locally gauge invariant. So, as a new rule we substitute the argument of the background fields by an invariant argument $\Delta x^\mu_{inv}$, defined as the line integral of the covariant derivatives of the original argument. As before, in order to obtain the theory physically equivalent to the original one, we add the Lagrange multiplier term. Using the local gauge freedom we fix the gauge taking $x^\mu(\xi) = x^\mu(\xi_0)$. The obtained gauge fixed action reduces to to the original action for the equations of motion for the Lagrange multiplier. The T-dual theory is obtained for the equations of motion for the gauge fields $v^\mu_\alpha$.

The generalized T-dualization procedure was investigated for a string moving in a weakly curved background composed of a constant metric, a linearly coordinate dependent Kalb-Ramond field with an infinitesimal filed strength and a constant dilaton field. It was first applied to all space-time coordinates in Ref. [6], and a T-dual was obtained. In Ref. [8], the procedure was applied to an arbitrary set of the initial coordinates. Choosing $d$ arbitrary directions, we denote $\mathcal{T}^a = \circ_{n=1}^{d} T^{\mu_n}$, $\mathcal{T}^i = \circ_{n=d+1}^{D} T^{\mu_n}$, and $\mathcal{T} = \circ_{n=1}^{D} T^{\mu_n}$, where $T^\mu$ stands for a T-dualization along direction $x^\mu$ and $\mathcal{T}_a = \circ_{n=1}^{d} T_{\mu_n}$, $\mathcal{T}_i = \circ_{n=d+1}^{D} T_{\mu_n}$, $\tilde{\mathcal{T}} = \circ_{n=1}^{D} T_{\mu_n}$, where $T_\mu$ stands for the T-dualization along a dual direction $y_\mu$. Performing the generalized procedure we proved the following composition laws:

$$\mathcal{T}^i \circ \mathcal{T}^a = \mathcal{T}, \quad \mathcal{T}_i \circ \mathcal{T}_a = \tilde{\mathcal{T}}, \quad \mathcal{T}_a \circ \mathcal{T}^a = 1, \tag{1}$$

where 1 denotes the identical transformation (T-dualization not performed). We found the explicit forms of the resulting theories and the corresponding T-dual coordinate transformation laws. These results complete the T-dualization diagram connecting all the theories T-dual to the initial theory.

$$\begin{tikzcd}
& S[x^i, y_a] \arrow[dl, shift left, "\mathcal{T}_a"] \arrow[dr, shift left, "\mathcal{T}^i"] & \\
S[x^\mu] \arrow[ur, shift left, "\mathcal{T}^a"] \arrow[rr, shift left, "\mathcal{T}"] & & S[y_\mu]. \arrow[ul, shift left, "\mathcal{T}_i"] \arrow[ll, shift left, "\tilde{\mathcal{T}}"]
\end{tikzcd}$$

The initial theory, describing the bosonic string moving in the weakly curved background is defined on the geometrical space. All its T-dual theories are non-geometric and non-local because they depend on variable $V^\mu$, which is a line integral of the derivatives of the dual coordinates. To all of these theories there corresponds a flux which is of the same type as the $R$ flux unlike the non-geometric theories with $Q$ flux, which have a local geometric description.

## 2. Bosonic string action

Let us consider the action [9, 10] describing the propagation of the bosonic string in a background composed of a space-time metric $G_{\mu\nu}$, a Kalb-Ramond field $B_{\mu\nu}$ and a dilaton field $\Phi$

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g} \Big[ \Big( \frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \Big) \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4\pi\kappa} \Phi(x) R^{(2)} \Big]. \tag{2}$$

The integration goes over two-dimensional world-sheet $\Sigma$ parametrized by $\xi^\alpha$ ($\xi^0 = \tau,\ \xi^1 = \sigma$), $g_{\alpha\beta}$ is intrinsic world-sheet metric, $R^{(2)}$ corresponding 2-dimensional scalar curvature, $x^\mu(\xi)$, $\mu = 0, 1, ..., D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$ and $\varepsilon^{01} = -1$.

In order to have a world-sheet conformal invariance on the quantum level, the background fields have to obey the space-time equations of motion which in the lowest order in slope parameter $\alpha'$, have the following form

$$
\begin{aligned}
&R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} + 2 D_\mu \partial_\nu \Phi = 0, \\
&D_\rho B^\rho{}_{\mu\nu} - 2\partial_\rho \Phi B^\rho{}_{\mu\nu} = 0, \\
&4(\partial\Phi)^2 - 4 D_\mu \partial^\mu \Phi + \frac{1}{12} B_{\mu\nu\rho} B^{\mu\nu\rho} - R + 4\pi\kappa \frac{D-26}{3} = 0\,,
\end{aligned} \tag{3}
$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and $D_\mu$ are Ricci tensor and covariant derivative with respect to space-time metric. We consider the weakly curved background, defined by

$$
G_{\mu\nu} = const, \quad B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho \equiv b_{\mu\nu} + h_{\mu\nu}(x), \quad \Phi = const. \tag{4}
$$

The Kalb-Ramond field strength $B_{\mu\nu\rho}$ is taken to be infinitesimal. All the calculations are done in the first order in $B_{\mu\nu\rho}$. In this approximation the weakly curved background is the solution of the space-time equations of motion (3).

Introducing the light-cone coordinates

$$
\xi^\pm = \frac{1}{2}(\tau \pm \sigma)
$$

and their derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$, taking a conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action (2) can be written as

$$
S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \tag{5}
$$

where

$$
\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x). \tag{6}
$$

## 3. The Generalized Buscher T-dualization procedure

The standard T-dualization procedure, enables one to find a T-dual of a given theory, applying the procedure to the coordinate directions which do not appear as the background field arguments. The generalized T-dualization procedure does not have this limitation. Both procedures are grounded in a localization of a global coordinate shift symmetry $\delta x^\mu = \lambda^\mu = const$. The first rule of the procedures is the introduction of the gauge fields $v^\mu_\alpha$ and the substitution of the ordinary derivatives with the covariant derivatives, defined by

$$
\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{7}
$$

If one imposes the following transformation law for the gauge fields

$$
\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau,\sigma)) \tag{8}
$$

one obtains $\delta D_\alpha x^\mu = 0$. In the case when the background does not depend on the coordinates, along which the T-dualization is performed, the first step is sufficient to obtain the gauge invariant action. However if the background depends on all the coordinates, an additional rule must be introduced. The new rule reads: *Substitute the background field argument (the* *coordinate $x^\mu$), by the invariant argument (invariant coordinate)*, defined as a line integral of the covariant derivatives of the original coordinate

$$\Delta x^\mu_{inv} \equiv \int_P d\xi^\alpha D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu, \quad \Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha. \tag{9}$$

The invariant coordinate is by definition nonlocal. The consequence of this will be a nonlocal T-dual theory, defined on the doubled geometrical space composed of the dual coordinate $y_\mu$ and its double $\tilde{y}_\mu$.

The common rule of the procedures is the addition of the Lagrange multiplier term which makes the introduced gauge fields nonphysical, by requiring that there field strength

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha \tag{10}$$

must be zero. This enables the physical equivalence of the theories. Following these rules we built the gauge invariant action.

The main object and the main crossway of the procedure are the gauge fixed action and their equations of motion, because for the equation of motion obtained varying the action over the Lagrange multipliers, one returns to the initial action. On the other hand for the equation of motion obtained varying the gauge fixed action over the gauge fields one obtains the T-dual theory. Comparing the solutions for the gauge fields in these two directions, one obtains the T-dual coordinate transformation laws. These laws are used in investigation of the relations between the non-commutativity characteristics of the spaces connected by T-duality.

The generalized procedure, can be generalized once more in order to allow the T-dualization of the backgrounds which do not have a global symmetry. The generalization was made in Ref. [7] for a bosonic string moving in a weakly curved background of the second order, which consists of the coordinate dependent metric and Kalb-Ramond field. One postulates the auxiliary action which inherits two important features of the gauge fixed action. It reduces to the initial theory for the equations of motion for the Lagrange multipliers and to the T-dual action for the equations of motion for the auxiliary fields.

### *3.1. Complete T-dualization*

If one applies the T-dualization procedure to all coordinates, one obtains a following gauge invariant action

$$S_{inv} = \kappa \int d^2\xi \Big[ D_+ x^\mu \Pi_{+\mu\nu}(\Delta x_{inv}) D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu)\Big], \tag{11}$$

which is physically equivalent to the initial action. Fixing the gauge by $x^\mu(\xi) = x^\mu(\xi_0)$, one obtains the gauge fixed action

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[ v^\mu_+ \Pi_{+\mu\nu}(\Delta V) v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu)\Big]. \tag{12}$$

In order to find a T-dual action one has to integrate out the gauge fields from (12).

The equations of motion with respect to the gauge fields $v^\mu_\pm$ are

$$\Pi_{\mp\mu\nu}(\Delta V) v^\nu_\pm + \frac{1}{2}\partial_\pm y_\mu = \mp \beta^\mp_\mu(V), \tag{13}$$

with the right hand side coming from the variation of the background fields argument, with $\beta^\pm_\mu(x) = \mp\frac{1}{2}h_{\mu\nu}[x]\partial_\mp x^\nu$. The equation of motion can be rewritten as

$$v^\mu_\pm(y) \quad = \quad -\kappa\, \Theta^{\mu\nu}_\pm[\Delta V(y)] \Big[ \partial_\pm y_\nu \pm 2\beta^\mp_\nu[V(y)]\Big], \tag{14}$$

where

$$\Theta_\pm^{\mu\nu}[\Delta V] = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu}[\Delta V] \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}[\Delta V], \tag{15}$$

and $G^E_{\mu\nu} \equiv [G - 4BG^{-1}B]_{\mu\nu}$, $\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ are the open string background fields: the effective metric and the non-commutativity parameter respectively. They are defined in analogy with the flat space-time open string background fields introduced in [11]. Tensors $\Pi_{\mp\mu\nu}$ and $\Theta_\pm^{\mu\nu}$ are connected by $\Theta_\pm^{\mu\nu}\Pi_{\mp\nu\rho} = \frac{1}{2\kappa}\delta^\mu_\rho$. Substituting (14) into the action (12), we obtain T-dual action

$$^\star S[y] \equiv S_{fix}[y] = \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu \Theta_-^{\mu\nu}[\Delta V^{(0)}(y)]\partial_- y_\nu, \tag{16}$$

where we neglected the term $\beta^-_\mu\beta^+_\nu$ as the infinitesimal of the second order, and the argument is given by

$$\Delta V^{(0)\mu}(y) = -\kappa\theta_0^{\mu\nu}\Delta y_\nu^{(0)} + (g^{-1})^{\mu\nu}\Delta\tilde{y}_\nu^{(0)}. \tag{17}$$

Comparing the initial action (5) with the T-dual action (16), we see that they are equal under following transformations $\partial_\pm x^\mu \to \partial_\pm y_\mu$ and $\Pi_{+\mu\nu}[x] \to \frac{\kappa}{2}\Theta_-^{\mu\nu}[\Delta V^{(0)}]$, which implies

$$\begin{aligned} G_{\mu\nu} &\to {}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}[\Delta V^{(0)}], \\ B_{\mu\nu}[x] &\to {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}[\Delta V^{(0)}], \end{aligned} \tag{18}$$

where $(G_E^{-1})^{\mu\nu}$ and $\theta^{\mu\nu}$ are introduced in (15).

The initial background consisted of a constant metric and a linearly coordinate dependent Kalb-Ramond field with an infinitesimal field strength. The T-dual background consists of coordinate dependent metric and Kalb-Ramond field, with the argument $\Delta V^\mu$, which is the linear combination of $y_\mu$ and its double $\tilde{y}_\mu$. Note that the variable $V^\mu$ and consequently T-dual action is not defined on the geometrical space (defined by the coordinate $y_\mu$) but on the so called doubled target space [12] composed of both $y_\mu$ and $\tilde{y}_\mu$.

### *3.2. Partial T-dualization*

If one choses only a subset of the initial coordinates, say $d$ coordinates $x^a$, and performs T-dualization procedure along these coordinates, one obtains the following gauge invariant action

$$\begin{aligned} S_{inv}[x^\mu, x^a_{inv}, y_a] &= \kappa\int d^2\xi\Big[\partial_+ x^i\Pi_{+ij}(x^i,\Delta x^a_{inv})\partial_- x^j \\ &+ \partial_+ x^i\Pi_{+ia}(x^i,\Delta x^a_{inv})D_- x^a + D_+ x^a\Pi_{+ai}(x^i,\Delta x^a_{inv})\partial_- x^i \\ &+ D_+ x^a\Pi_{+ab}(x^i,\Delta x^a_{inv})D_- x^b + \frac{1}{2}(v^a_+\partial_- y_a - v^a_-\partial_+ y_a)\Big]. \end{aligned} \tag{19}$$

This action is obtained localizing the global shift symmetry only for the coordinates $x^a$, by introducing the gauge fields $v^a_\alpha$. The ordinary derivatives $\partial_\alpha x^a$ were substituted by the covariant derivatives $D_\alpha x^a = \partial_\alpha x^a + v^a_\alpha$. The covariant derivatives are invariant under the standard gauge transformations $\delta v^a_\alpha = -\partial_\alpha\lambda^a$. The coordinates $x^a$ in the argument of the background fields were substituted by their invariant extension, defined by $\Delta x^a_{inv} \equiv \int_P d\xi^\alpha D_\alpha x^a = x^a - x^a(\xi_0) + \Delta V^a$, where $\Delta V^a \equiv \int_P d\xi^\alpha v^a_\alpha$. The physical equivalence is preserved by adding the Lagrange multiplier term (the last term in the action). Fixing the gauge by $x^a(\xi) = x^a(\xi_0)$ one obtains the gauge fixed action

$$\begin{aligned} S_{fix}[x^i, v_\pm^a, y_a] &= \kappa \int d^2\xi \Big[\partial_+ x^i \Pi_{+ij}(x^i, \Delta V^a)\partial_- x^j \\ &+ \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a) v_-^a + v_+^a \Pi_{+ai}(x^i, \Delta V^a)\partial_- x^i \\ &+ v_+^a \Pi_{+ab}(x^i, \Delta V^a) v_-^b + \frac{1}{2}(v_+^a \partial_- y_a - v_-^a \partial_+ y_a)\Big]. \end{aligned} \tag{20}$$

This action reduces to the initial one for the equations of motion obtained varying over the Lagrange multipliers. The T-dual action is obtained for the equations of motion for the gauge fields. It reads

$$\begin{aligned} S[x^i, y_a] &= \kappa \int d^2\xi \Big[\partial_+ x^i \overline{\Pi}_{+ij}(x^i, \Delta V^a(x^i, y_a))\partial_- x^j \\ &-\kappa\, \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a(x^i, y_a))\tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a))\partial_- y_b \\ &+\kappa\, \partial_+ y_a \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a))\Pi_{+bi}(x^i, \Delta V^a(x^i, y_a))\partial_- x^i \\ &+\frac{\kappa}{2}\, \partial_+ y_a \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a))\partial_- y_b\Big]. \end{aligned} \tag{21}$$

The T-dual background fields compositions are the inverses of the already known background compositions, divided into two coordinate subspaces, the subspace formed by the coordinates we T-dualize and the subspace formed by the rest of the coordinates. The background field compositions $\overline{\Pi}_{\pm ij}$ and $\tilde{\Theta}_\pm^{ab}$ are defined as the inverses of the background field compositions $\Theta_\mp^{jk}$ and $\Pi_{\mp bc}$, which are the parts of $\Theta_\mp^{\mu\nu}$ and $\Pi_{\mp\mu\nu}$ in an appropriate subspace

$$\begin{aligned} \overline{\Pi}_{\pm ij}\Theta_\mp^{jk} &= \Theta_\mp^{kj}\overline{\Pi}_{\pm ji} = \frac{1}{2\kappa}\delta_i^k, \\ \tilde{\Theta}_\pm^{ab}\Pi_{\mp bc} &= \Pi_{\mp cb}\tilde{\Theta}_\pm^{ba} = \frac{1}{2\kappa}\delta_c^a. \end{aligned} \tag{22}$$

It can be shown that

$$\overline{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa \Pi_{+ia}\tilde{\Theta}_-^{ab}\Pi_{+bj}. \tag{23}$$

The argument of the background fields is

$$\begin{aligned} \Delta V^{(0)a}(x^i, y_a) &= -\kappa\Big[\tilde{\Theta}_{0+}^{ab}\Pi_{0-bi} + \tilde{\Theta}_{0-}^{ab}\Pi_{0+bi}\Big]\Delta x^{(0)i} \\ &- \kappa\Big[\tilde{\Theta}_{0+}^{ab}\Pi_{0-bi} - \tilde{\Theta}_{0-}^{ab}\Pi_{0+bi}\Big]\Delta\tilde{x}^{(0)i} \\ &- \frac{\kappa}{2}\Big[\tilde{\Theta}_{0+}^{ab} + \tilde{\Theta}_{0-}^{ab}\Big]\Delta y_b^{(0)} - \frac{\kappa}{2}\Big[\tilde{\Theta}_{0+}^{ab} - \tilde{\Theta}_{0-}^{ab}\Big]\Delta\tilde{y}_b^{(0)}. \end{aligned} \tag{24}$$

Calculating the symmetric and antisymmetric part of the background fields we obtain a T-dual metric and a T-dual Kalb-Ramond field

$$\begin{aligned} &\bullet G_{ij} = \overline{G}_{ij} = G_{ij} - G_{ia}(\tilde{G}_E^{-1})^{ab}G_{bj} \\ &\quad -2\kappa\Big(B_{ia}\tilde{\theta}^{ab}G_{bj} + G_{ia}\tilde{\theta}^{ab}B_{bj}\Big) - 4B_{ia}(\tilde{G}_E^{-1})^{ab}B_{bj}, \\ &\bullet B_{ij} = \overline{B}_{ij} = B_{ij} - \frac{\kappa}{2}G_{ia}\tilde{\theta}^{ab}G_{bj} - B_{ia}(\tilde{G}_E^{-1})^{ab}G_{bj} \\ &\quad -G_{ia}(\tilde{G}_E^{-1})^{ab}B_{bj} - 2\kappa B_{ia}\tilde{\theta}^{ab}B_{bj}, \\ &\bullet G^{ab} = (\tilde{G}_E^{-1})^{ab}, \\ &\bullet B^{ab} = \frac{\kappa}{2}\tilde{\theta}^{ab}, \\ &\bullet G^a{}_i = \kappa\tilde{\theta}^{ab}G_{bi} + 2(\tilde{G}_E^{-1})^{ab}B_{bi}, \\ &\bullet B^a{}_i = \kappa\tilde{\theta}^{ab}B_{bi} + \frac{1}{2}(\tilde{G}_E^{-1})^{ab}G_{bi}. \end{aligned} \tag{25}$$

As the constituents of the T-dual background field there appear the effective metric in the subspace $a$, defined by $\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd}B_{db}$, the non-commutativity parameter in the same subspace $\tilde{\theta}^{ab} \equiv -\frac{2}{\kappa}(\tilde{G}_E^{-1})^{ac}B_{cd}(\tilde{G}^{-1})^{db}$, which combined give the new theta function $\tilde{\Theta}^{ab}_{\pm} = \tilde{\theta}^{ab} \mp \frac{1}{\kappa}(\tilde{G}_E^{-1})^{ab}$.

## 4. Open string T-dualization

In paper [13] we investigated a T-duality of an open string moving in a weakly curved background. The open string moving in a weakly curved background was a subject of investigation in our papers [14, 15, 16]. Solving the boundary conditions at the open string end-points, one obtains the effective closed string described by the effective closed string theory $S^{eff}$, defined on the doubled space $(q^\mu, \tilde{q}^\mu)$. As the effective theory is closed string theory, one can try to apply the generalized T-dualization procedure to this theory. The effective theory is defined on the doubled theory, just as the T-duals of the closed string theory moving in the weakly curved background. So, the application in this case resembles the application of the T-dualization procedure to the T-dual theories.

The effective theory of the open string moving in the weakly curved background, obtained for the solution of the boundary conditions equals

$$S^{eff} = \kappa \int d\tau \int_{-\pi}^{\pi} d\sigma\, \partial_+ q^\mu\, \Pi^{eff}_{+\mu\nu}(q, 2b\tilde{q})\, \partial_- q^\nu, \tag{26}$$

where

$$\Pi^{eff}_{\pm\mu\nu}(q, 2b\tilde{q}) \equiv B^{eff}_{\mu\nu}(2b\tilde{q}) \pm \frac{1}{2} G^{eff}_{\mu\nu}(q). \tag{27}$$

The effective variable is $q^\mu(\sigma)$, an even part of the initial coordinate. The effective metric and the Kalb-Ramond field are explicitly given by

$$\begin{aligned} G^{eff}_{\mu\nu}(q) &= G^E_{\mu\nu}(q) := (G - 4B^2(q))_{\mu\nu}, \\ B^{eff}_{\mu\nu}(2b\tilde{q}) &= -\frac{\kappa}{2}\Big(g_E \Delta\theta(2b\tilde{q}) g_E\Big)_{\mu\nu}, \end{aligned} \tag{28}$$

where $\Delta\theta^{\mu\nu}$ is the infinitesimal part of the non-commutativity parameter $\theta^{\mu\nu} = -\frac{2}{\kappa}\Big[G_E^{-1} B G^{-1}\Big]^{\mu\nu} = \theta_0^{\mu\nu} - \frac{2}{\kappa}\Big[g_E^{-1}(h + 4bhb)g_E^{-1}\Big]^{\mu\nu}$. In paper [13] we applied the generalized Buscher T-dualization procedure, to the effective theory along all effective directions $q^\mu$. Following the procedure we find the gauge fixed action

$$\mathcal{S}_{fix} = \kappa \int d^2\xi \Big[v^\mu_+ \Pi^{eff}_{+\mu\nu}(\Delta V, 2b\Delta\tilde{V}) v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- \varrho_\mu - v^\mu_- \partial_+ \varrho_\mu)\Big], \tag{29}$$

obtained from the effective action (26), by substituting the light-cone derivatives $\partial_\pm q^\mu$ with the covariant derivatives $D_\pm q^\mu = \partial_\pm q^\mu + v^\mu_\pm$, where $v^\mu_\pm$ are the gauge fields, which transform as $\delta v^\mu_\pm = -\partial_\pm \lambda^\mu$. The argument of the background fields is substituted with an invariant argument, which is obtained substituting the effective coordinate $q^\mu$ and its double $\tilde{q}^\mu$ with an invariant effective coordinate and its double, defined by the following line integrals of the gauge fields $\Delta V^\mu = \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-)$, and $\Delta\tilde{V}^\mu = \int_P (d\xi^+ v^\mu_+ - d\xi^- v^\mu_-)$. The physical equivalence was achieved by adding the Lagrange multiplier term $\frac{1}{2}(v^\mu_+ \partial_- \varrho_\mu - v^\mu_- \partial_+ \varrho_\mu)$ and the gauge is fixed with $q^\mu(\xi) = q^\mu(\xi_0)$.

The T-dual theory was obtained for the equation of motion for the gauge fields. The T-dual action reads

$$^\star\mathcal{S} = \kappa \int d^2\xi \partial_+ \varrho_\mu \frac{\kappa}{2} (\Theta^{eff}_-)^{\mu\nu} (\Delta V(\varrho), 2b\Delta\tilde{V}(\varrho)) \partial_- \varrho_\nu, \tag{30}$$

where

$$(\Theta^{eff}_{\pm})^{\mu\nu}(x,y) \equiv \Theta^{\mu\nu}_{\pm}(G_{eff}(x), B_{eff}(y)) = \theta^{\mu\nu}_{eff}(y) \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}(x), \tag{31}$$

$\theta^{\mu\nu}_{eff} := \theta^{\mu\nu}(G_{eff}(x), B_{eff}(y)) = -\frac{2}{\kappa}\Big(G_E^{-1}(G_{eff}(x), B_{eff}(y))B_{eff}(y)G^{-1}_{eff}(x)\Big)^{\mu\nu}$ and the argument is

$$\begin{aligned} V_0^{\mu}(\varrho) &= (g_E^{-1})^{\mu\nu}(G^{eff}, B^{eff})\tilde{\varrho}_\nu = (g_E^{-1})^{\mu\nu}\tilde{\varrho}_\nu, \\ \tilde{V}_0^{\mu}(\varrho) &= (g_E^{-1})^{\mu\nu}(G^{eff}, B^{eff})\varrho_\nu = (g_E^{-1})^{\mu\nu}\varrho_\nu. \end{aligned} \tag{32}$$

The T-dual metric ${}^\star G^{\mu\nu}$ which depends on the first variable $\Delta V^\mu$ and the T-dual Kalb-Ramond field ${}^\star B^{\mu\nu}$, which depends on the second variable $2b^\mu{}_\nu \Delta\tilde{V}^\nu$ are

$$\begin{aligned} {}^\star G^{\mu\nu} &= (G_E^{-1})^{\mu\nu}(\Delta V), \\ {}^\star B^{\mu\nu} &= \frac{\kappa}{2}(\theta^{eff})^{\mu\nu}(2b\Delta\tilde{V}) = \frac{\kappa}{2}\Delta\theta^{\mu\nu}(2b\Delta\tilde{V}). \end{aligned} \tag{33}$$

We see, that the effective metric has transformed to its inverse and that the Kalb-Ramond field has transformed to the infinitesimal part of the non-commutativity parameter.

Finally, we searched for the open string theory $\tilde{S}$ such that its effective theory is ${}^\star S^{eff}$ exactly. We found

$$\widetilde{S}[y] = \kappa \int_\Sigma d^2\xi\ \partial_+ y_\mu \widetilde{\Pi}^{\mu\nu}_+(y)\partial_- y_\nu, \tag{34}$$

with

$$\begin{aligned} \widetilde{G} &= -(C^T)^{-1}GC^{-1}, \\ \widetilde{B}(y) &= \pm(C^T)^{-1}(b - h(C^{-1}y))C^{-1}, \end{aligned} \tag{35}$$

where $C$ makes a connection between the variables of the effective theory of $\tilde{S}$ and the T-dual theory (30)

$$\begin{aligned} q_\mu(y) &= C_{\mu\nu}(g_E^{-1})^{\nu\rho}\tilde{\varrho}_\rho\,, \\ \bar{q}_\mu(y) &= \mp C_{\mu\nu}2(G^{-1}bg_E^{-1})^{\nu\rho}\varrho_\rho. \end{aligned} \tag{36}$$

In the closed string moving in the weakly curved background case, the T-duality transforms the geometrical background into a doubled non-geometrical background. It transforms a constant metric to a coordinate dependent effective metric inverse, while the linearly coordinate dependent Kalb-Ramond field is transformed into a coordinate dependent non-commutativity parameter. In the open string case, the T-dual theory remains geometric. T-duality transforms the constant metric of the weakly curved background to a constant T-dual metric, while the coordinate dependent Kalb-Ramond field transforms again to the coordinate dependent field.

In paper [17] a generalization of the standard analysis of the open bosonic string moving in a flat background is addressed. The T-dualization was performed in two ways, first in terms of non-constant vector fields in which case the Buscher T-dualization procedure can not be applied and second in terms of the field strengths of the gauge fields. The role of the gauge fields, which live on the string boundary, is to restore the symmetries of the closed string: the local gauge symmetry of the Kalb-Ramond field and the general coordinate transformations, at the string end-points. The investigation lead to a discovery of the geometrical features of the non-geometry.

## Conclusion

The generalized T-dualization procedure, enabled T-dualization over the non isometry directions. It gives the new insights into a connection between the spaces connected by T-duality. It enabled further investigations of the closed string non-commutativity [18]. Comparing the solutions for the gauge fields which transform the gauge fixed actions into the initial or the T-dual actions, one obtains the T-dual coordinate transformation laws. Using these laws one can find how does for example a standard Poisson bracket transform. It is obtained that the original theory which is commutative is equivalent to the non-commutative T-dual theory, whose Poisson brackets are proportional to the background fluxes times winding and momentum numbers. The obtained results add novelty to the form and the origin of different non-commutative structures.

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