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SCALE TRANSFORMATIONS IN PHASE SPACE AND STRETCHED STATES OF A HARMONIC OSCILLATOR

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We consider scale transformations $(q,p) \to (\lambda q, \lambda p)$ in phase space. They induce transformations of the Husimi functions $H(q,p)$ defined in this space. We consider the Husimi functions for states that are arbitrary superpositions of n-particle states of a harmonic oscillator. We develop a method that allows finding so-called stretched states to which these superpositions transform under such a scale transformation. We study the properties of the stretched states and calculate their density matrices in explicit form. We establish that the density matrix structure can be described using negative binomial distributions. We find expressions for the energy and entropy of stretched states and calculate the means of the number-of-states operator. We give the form of the Heisenberg and Robertson–Schrödinger uncertainty relations for stretched states.

Keywords: phase space, Husimi function, scale transformation, harmonic oscillator, stretched state, uncertainty relation

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1. Introduction

We propose a new method for constructing the quantum states arising in certain physical processes. This method is based on using quasiprobability distributions to describe quantum states. These distributions are defined on the phase space, and physical processes are associated with some transformations of this space. Phase-space transformations induce transformations of the functions defined on them. Determining the physical states corresponding to the transformed quasiprobability distributions, we can find the result

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of the action of a physical process on the initial quantum state. This is the general scheme of our approach. Here, we use Husimi functions as quasiprobability distributions and consider the amplification of quantum states as a physical process. This process can be associated with a scale transformation in the phase space.

It is known that quantum mechanics admits several mutually equivalent formulations. Moreover, one of them can be more convenient for a particular problem, which justifies their joint existence and study. The most popular is the formulation where a quantum state is associated with a vector in a Hilbert space and observable quantities are associated with operators acting in this space. Another also very popular formulation is based on using so-called quasiprobability distributions in phase space. Historically, this approach is related to attempts to use the similarity between the statistical nature of quantum phenomena and classical statistical processes.

In classical statistical mechanics, a physical state is associated with some distribution function $\rho(q,p)$ defined on the phase space of a system. This function is the probability density for the system to be in the state characterized by the parameters q and p. The arguments are the coordinates and momenta, where q is exactly the coordinate of the point in space where the system is at a given instant and p is exactly the momentum which the system has at this instant. Moreover, we assume that the coordinates and momenta can be measured simultaneously. Knowing the distribution function for a system, we can calculate its various characteristics, for instance, the means of certain quantities. For this, a physical quantity is also associated with a corresponding function $F(q,p)$ defined on the phase space, and to find its mean in the state with a distribution function $\rho(q,p)$, we must calculate the integral

$$\langle F\rangle = \int F(q,p)\rho(q,p)\,dq\,dp. \tag{1}$$

From the very beginning of quantum mechanics, there were attempts to regard it as a statistical theory and develop a formalism similar to classical statistical theory, namely, to associate a quantum state with some function $D(q,p)$ that is defined on the phase space and uniquely characterizes the state. Each operator $\hat{A}$ related to an observable quantity can be associated with the function $A_D(q,p)$, which is also defined over the whole phase space such that the mean $\langle\hat{A}\rangle$ of the operator $\hat{A}$ in the state defined by a quasiprobability distribution $D(q,p)$ can be calculated as the integral

$$\langle\hat{A}\rangle = \int A_D(q,p)D_\rho(q,p)\,dq\,dp. \tag{2}$$

The function $D(q,p)$ is called the quasiprobability distribution associated with a given quantum state, and the function $A_D(q,p)$ is the $\hat{A}$-operator symbol, constructed according to the given quasiprobability distribution $D(q,p)$.

There is a set of quasiprobability states in quantum mechanics, and each operator $\hat{A}$ therefore has several symbols. The Wigner $W(q,p)$ [1], Husimi–Kano $Q(q,p)$ [2], [3], and Glauber–Sudarshan $P(q,p)$ [4], [5] functions are the most famous of them. General properties of quasiprobability distributions were studied in [6]–[9], where methods for constructing operator symbols associated with these quasiprobability distributions were also developed. Problems of applying them in quantum optics were considered in [10]–[14].

In addition to various quasiprobability distributions, there is a true probability distribution in quantum mechanics defined on the phase space and completely determining the quantum state. It is called the symplectic tomogram of a quantum state [15]. Properties of tomograms were discussed in [16].

Here, we deal with the Husimi–Kano function $Q(q,p)$, which we call the Husimi function for brevity. Its definition and general properties are presented in Sec. 2. The principal idea of our approach is as follows. There is a set of quantum states for which the exact analytic form of the Husimi functions is known. We can consider a transformation of the phase space (q,p) that induces a transformation of the Husimi functions. As a result, new functions arise that depend on the transformation parameters. We can try to represent these new functions as a sum of already known Husimi functions with coefficients depending on the parameters of the phase-space transformation. Using this sum of Husimi functions, we can then find the density matrix of the transformed state. In several cases, the phase-space transformation can be related to a particular physical process.

We consider a scale transformation in the phase space of the form

$$(q,p) \to (\lambda q, \lambda p), \quad |\lambda|^2 \leq 1. \tag{3}$$

It was proved in [17] that if $Q(q,p)$ is a Husimi function of a quantum state and $\lambda < 1$, then

$$Q_\lambda(q,p) = \lambda^2 Q(\lambda q, \lambda p) \tag{4}$$

is also a Husimi function for some quantum state.

In Sec. 3, we show that transformation (4) of the Husimi function naturally arises in quantum optics problems. This stimulates our interest in this problem. We study what happens with the states of a harmonic oscillator under such a transformation.

In Sec. 4, we consider the pure state that is an arbitrary superposition of n-particle states. We show that a pure state becomes a mixed state as a result of this transformation, and we find its density matrix. This mixed λ state contains an infinite set of pure states, and the probabilities with which these pure states are included into a mixed state form a negative binomial distribution.

In the case where we deal with an initial single N-particle state, the transformed λ state contains all M-particle states with $M \geq N$. Moreover, the less the parameter λ^2 is, the smoother the distribution of the given pure states in the mixed state. Roughly speaking, we can assume that under λ transformation (3), states with $M > N$ arise from the state $|N\rangle$. We call such mixed states stretched states. In Sec. 5, we find the means of the particle number operator for stretched states. In Sec. 6, we calculate the entropy of the von Neumann stretched states. In Sec. 7, we find the forms of the Heisenberg and Robertson–Schrödinger uncertainty relations for stretched states. For such states, we show that the factor λ^{-4} appears in the right-hand side of the uncertainty relations, i.e., the uncertainty of states increases under λ transformation (3). We briefly discuss the possible physical effects of this fact.

Everywhere in this paper, we assume that $|\lambda| < 1$. But the question of what happens and what states can appear for $|\lambda| > 1$ arises. In Sec. 8, we formally apply the argumentation scheme used for $|\lambda| < 1$ to the case $|\lambda| > 1$. We show that in this case, the Husimi functions of n-particle states of the harmonic oscillator are transformed into quantities that are not the Husimi functions of any quantum states.

2. Husimi functions of harmonic-oscillator states

We consider the one-dimensional harmonic oscillator. Its Hamiltonian $\widehat{H}$ is defined as

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + \frac{\omega^2}{2}x^2 = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{5}$$

where the coordinate and momentum operators $\hat{q}$ and $\hat{p}$ and also the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ are

$$\begin{aligned} &\hat{p} = -i\hbar\frac{d}{dx}, \qquad \hat{q} = x, \\ &\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\hat{p} + i\omega\hat{q}), \qquad \hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\hat{p} - i\omega\hat{q}). \end{aligned} \tag{6}$$

For the Hamiltonian H, the n-particle states $|n\rangle$ are its eigenfunctions:

$$\widehat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle. \tag{7}$$

The coherent states of the harmonic oscillator are

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \tag{8}$$

Here, α is an arbitrary complex number.

Let there be a quantum state defined by the density operator $\hat{\rho}$. Using coherent states (8), we can then construct its Husimi function:

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int \langle\alpha|x\rangle\rho(x, y)\langle y|\alpha\rangle\, dx\, dy. \tag{9}$$

If the quantum state is pure and is described by the wave function $|\psi\rangle$, then its Husimi function is

$$Q(q, p) = \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle. \tag{10}$$

The Husimi function is defined on the phase space with the coordinates (q, p). The scale transformation $(q, p) \to (\lambda q, \lambda p)$ in this space was considered in [17]. It was shown that if $Q(q, p)$ is a Husimi function of a quantum state, then $\lambda^2 Q(\lambda q, \lambda p)$ is also a Husimi function of some quantum state if $|\lambda|^2 \leq 1$.

Below, we show that transformation (3) can be associated with certain physical processes, for instance, with the state of an electromagnetic field passing through a quantum amplifier [18], [19]. Therefore, the problem of constructing an explicit form of such transformations for a particular state and studying their properties is relevant. In several cases, this problem can be solved exactly.

Here, we consider a harmonic oscillator and find the density matrices of those states into which the superpositions of its n-particle states are mapped under scale transformation (3). For this, we use a special method based on some properties of Husimi functions of such states.

3. Connection of scale transformations with quantum optics problems

Before proceeding to a systematic development of the formalism, we show the relation of this approach to quantum optics problems and explain how scale transformation (3) appears in such a formulation of the problem. The general idea can be understood using an example of a simple linear light amplifier consisting of partially inverted two-level atoms. The resonance Hamiltonian for the interaction between the field and the atoms is

$$\widehat{H} = \hbar k \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix}. \tag{11}$$

This is an interaction Hamiltonian in the Jaynes–Cummings model. It has several interesting properties, in particular, supersymmetry [20]. For this Hamiltonian, the equation for the density matrix $\hat{\rho}$ of an electromagnetic field can be written in the first approximation as [13]

$$\frac{\partial\hat{\rho}}{\partial t} = -kN_1(\hat{a}\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger) - kN_2(\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}). \tag{12}$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the electromagnetic field, N_1 and N_2 are the populations of the upper and lower levels of a two-level atoms, and k is the amplification coefficient.

Using relation (9) between the density matrix and the Husimi function, we can pass from operator equation (12) to an ordinary differential equation for the Husimi function. Using this equation, the expression for the Husimi function for the state at the exit from a quantum amplifier was obtained in [18]:

$$Q_{\rm out}(\alpha,t)=\frac{1}{G^2}Q_{\rm in}\left(\frac{\alpha}{G}\right)=\left\langle\frac{\alpha}{G}\middle|\hat\rho_{\rm in}\middle|\frac{\alpha}{G}\right\rangle, \tag{13}$$

where

$$G(t)=e^{2(N_1-N_2)kt}. \tag{14}$$

We see that expression (13) coincides with (4) at $\lambda=G^{-1}$. Hence, scale transformation (3) in the phase space turns out to be related to the action of a quantum amplifier, and the form of this transformation is defined by the structure of Hamiltonian (11). Accordingly, the action of the amplifier on an arbitrary quantum state can be described using the scale transformation in the phase space.

Formally, we here deal with only the states of a harmonic oscillator, but keeping in mind that the method developed here is assumed to be applicable to quantum optics problems, we sometimes call these states n-photon or n-particle Fock states.

4. Density matrices of stretched states

We first consider the N-particle state $|N\rangle$. Its Husimi function is

$$Q_N(q,p)=\langle\alpha|N\rangle\langle N|\alpha\rangle=e^{-|\alpha|^2}\frac{|\alpha|^{2N}}{N!}. \tag{15}$$

After scale transformation (3), we have

$$Q_N^\lambda(q,p)=\lambda^2e^{-\lambda^2|\alpha|^2}\frac{\lambda^{2N}|\alpha|^{2N}}{N!}. \tag{16}$$

We want to represent expression (16) as a sum of Husimi functions $Q_j(q,p)$ with different j. For this, we write it as

$$Q_N^\lambda(q,p)=e^{-\lambda^2|\alpha|^2}\frac{\lambda^{2N+2}|\alpha|^{2N}}{N!}=e^{-|\alpha|^2}e^{(1-\lambda^2)|\alpha|^2}\frac{\lambda^{2N+2}|\alpha|^{2N}}{N!}. \tag{17}$$

Expanding the exponential $e^{(1-\lambda^2)|\alpha|^2}$ in a series, we now obtain

$$\begin{aligned}Q_N^\lambda(q,p)&=e^{-|\alpha|^2}\sum_{j=0}^{\infty}\frac{(1-\lambda^2)^j|\alpha|^{2j}}{j!}\frac{\lambda^{2N+2}|\alpha|^{2N}}{N!}=\\&=\sum_{j=0}^{\infty}\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!\,N!}e^{-|\alpha|^2}\frac{|\alpha|^{2(N+j)}}{(N+j)!}=\\&=\sum_{j=0}^{\infty}\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!\,N!}Q_{N+j}(q,p).\end{aligned} \tag{18}$$

Passing from the Husimi function to the density matrix, we find that state (16) corresponds to the density matrix

$$\hat\rho_N^\lambda=\sum_{j=0}^{\infty}\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!\,N!}|N+j\rangle\langle N+j|. \tag{19}$$

This density matrix has a diagonal form with the first N diagonal elements (with the labels $0, 1, \ldots, N-1$) equal to zero and the other main diagonal elements

$$F_{N+j}^{N} = \frac{(1-\lambda^2)^j (N+j)!}{j!\,N!}\lambda^{2N+2}, \quad j = 0, 1, \ldots. \tag{20}$$

Quantities (20) form a negative binomial distribution. The elements of this distribution are given by [21]

$$f(k, r, p) = \binom{r+k-1}{k} p^r q^k = \frac{(r+k-1)!}{(r-1)!\,k!} p^r (1-p)^k, \quad k = 0, 1, 2, \ldots. \tag{21}$$

They are determined by two parameters r and p, $q = 1-p$, and k is the element label in the distribution. In our case, $r = N+1$, $k = j$, and $p = \lambda^2$. We therefore have

$$F_{N+j}^{N} = f(j, N+1, \lambda^2). \tag{22}$$

We now consider a state that is a superposition of two k-particle states $|M\rangle$ and $|N\rangle$. Its wave function and density matrix are

$$\begin{gathered} \psi_{M,N} = c_M|M\rangle + c_N|N\rangle, \qquad |c_M|^2 + |c_N|^2 = 1, \\ \hat{\rho}_{M,N} = (c_M|M\rangle + c_N|N\rangle)(c_M^*\langle M| + c_N^*\langle N|). \end{gathered} \tag{23}$$

We find the density matrix of the stretched state $\hat{\rho}_{M,N}^{\lambda}$. The Husimi function of state (23) can be written as

$$\begin{aligned} Q_{M,N}(\alpha) &= \langle\alpha|\rho_{M,N}|\alpha\rangle = \\ &= e^{-|\alpha|^2/2}\left(c_M\frac{(\alpha^*)^M}{\sqrt{M!}} + c_N\frac{(\alpha^*)^N}{\sqrt{N!}}\right)e^{-|\alpha|^2/2}\left(c_M^*\frac{\alpha^M}{\sqrt{M!}} + c_N^*\frac{\alpha^N}{\sqrt{N!}}\right). \end{aligned} \tag{24}$$

After transformation (3), we have

$$\begin{aligned} Q_{M,N}^{\lambda}(\lambda q, \lambda p) &= \lambda^2 e^{-\lambda^2|\alpha|^2}\left(c_M\frac{\lambda^M(\alpha^*)^M}{\sqrt{M!}} + c_N\frac{\lambda^N(\alpha^*)^N}{\sqrt{N!}}\right)\left(c_M^*\frac{\lambda^M\alpha^M}{\sqrt{M!}} + c_N^*\frac{\lambda^N\alpha^N}{\sqrt{N!}}\right) = \\ &= e^{-|\alpha|^2}\lambda^2 e^{(1-\lambda^2)|\alpha|^2}\left(c_M\frac{\lambda^M(\alpha^*)^M}{\sqrt{M!}} + c_N\frac{\lambda^N(\alpha^*)^N}{\sqrt{N!}}\right)\left(c_M^*\frac{\lambda^M\alpha^M}{\sqrt{M!}} + c_N^*\frac{\lambda^N\alpha^N}{\sqrt{N!}}\right). \end{aligned} \tag{25}$$

As in the previous case, we expand the exponential $e^{(1-\lambda^2)|\alpha|^2}$ in a series, and Husimi function (25) then becomes

$$Q_{M,N}^{\lambda}(\lambda q, \lambda p) = e^{-|\alpha|^2}\lambda^2\sum_{j=0}^{\infty}\frac{(1-\lambda^2)^j|\alpha|^{2j}}{j!}\left(c_M\frac{\lambda^M(\alpha^*)^M}{\sqrt{M!}} + c_N\frac{\lambda^N(\alpha^*)^N}{\sqrt{N!}}\right)\left(c_M^*\frac{\lambda^M\alpha^M}{\sqrt{M!}} + c_N^*\frac{\lambda^N\alpha^N}{\sqrt{N!}}\right).$$

We now take the relations

$$e^{-|\alpha|^2/2}\frac{\alpha^{s+j}}{\sqrt{(s+j)!}} = \langle s+j|\alpha\rangle, \qquad e^{-|\alpha|^2/2}\frac{(\alpha^*)^{k+j}}{\sqrt{(k+j)!}} = \langle\alpha|k+j\rangle \tag{26}$$

into account. Using them, we obtain the expression for Husimi function (25):

$$\begin{aligned} Q_{N,M}^{\lambda}(\lambda q, \lambda p) = \sum_{j=0}^{\infty}\frac{\lambda^2(1-\lambda^2)^j}{j!}\langle\alpha|\left(\sqrt{\frac{(M+j)!}{M!}}\lambda^M c_M|M+j\rangle + \sqrt{\frac{(N+j)!}{N!}}\lambda^N c_N|N+j\rangle\right) \times \\ \times\left(\sqrt{\frac{(M+j)!}{M!}}\lambda^M c_M^*\langle M+j| + \sqrt{\frac{(N+j)!}{N!}}\lambda^N c_N^*\langle N+j|\right)|\alpha\rangle. \end{aligned} \tag{27}$$

It hence follows that the density matrix of the stretched state $\hat{\rho}^{\lambda}_{M,N}$ is

$$\hat{\rho}^{\lambda}_{N,M} = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!} \left(\sqrt{\frac{(M+j)!}{M!}} \lambda^M c_M |M+j\rangle + \sqrt{\frac{(N+j)!}{N!}} \lambda^N c_N |N+j\rangle \right) \times$$
$$\times \left(\sqrt{\frac{(M+j)!}{M!}} \lambda^M c_M^* \langle M+j| + \sqrt{\frac{(N+j)!}{N!}} \lambda^N c_N^* \langle N+j| \right). \tag{28}$$

We consider the structure of this matrix.

Density matrix (28) has three nonzero diagonals. Let $M = N + k$. Then the first N elements with the labels $0, 1, \ldots, N-1$ on the main diagonal are equal to zero. The next k diagonal elements with the labels $N, \ldots, N+k-1$ coincide with the first k nonzero diagonal elements of density matrix (19),

$$D_{N+j,N+j} = \frac{(1-\lambda^2)^j (N+j)!}{j!\,N!} \lambda^{2+2N} |c_N|^2, \quad j = 0, 1, \ldots, k-1. \tag{29}$$

The remaining elements on the main diagonal of density matrix (28) have the form of the sum of the diagonal elements of the density matrices ρ^{λ}_N and ρ^{λ}_M, which are given by formula (19):

$$D_{N+k+j,N+k+j} = \frac{(1-\lambda^2)^{k+j} (N+k+j)!}{(k+j)!N!} \lambda^{2+2N} |c_N|^2 +$$
$$+ \frac{(1-\lambda^2)^j (M+j)!}{j!M!} \lambda^{2+2M} |c_M|^2, \quad j = 0, 1, \ldots. \tag{30}$$

Therefore, the main diagonal of the density matrix $\hat{\rho}^{\lambda}_{N,M}$ given by (28) has the form of the sum of the main diagonals of the density matrices $\hat{\rho}^{\lambda}_N$ and $\hat{\rho}^{\lambda}_M$ multiplied by the coefficients $|c_N|^2$ and $|c_M|^2$.

In addition to the main diagonal, density matrix (28) has nonzero elements on two more diagonals, above and below the main diagonal at a distance of k steps. These nonzero elements are located with the coordinates $(N+i, N+k+i)$ and $(N+k+i, N+i)$, $i = 0, 1, \ldots$. The elements of the density matrix with the coordinates $(N+j, M+j)$ and $(M+j, N+j)$ have the forms

$$\begin{aligned} D_{N+j,M+j} &= \frac{(1-\lambda^2)^j}{j!} \lambda^{N+M+2} \sqrt{\frac{(N+j)!\,(M+j)!}{N!\,M!}} c_N c_M^*, \\ D_{M+j,N+j} &= \frac{(1-\lambda^2)^j}{j!} \lambda^{N+M+2} \sqrt{\frac{(N+j)!\,(M+j)!}{N!\,M!}} c_N^* c_M, \end{aligned} \qquad j = 0, 1, \ldots. \tag{31}$$

We can now consider a general case of an arbitrary state of the harmonic oscillator. The strategy remains the same as in the case of state (19), which is a superposition of two Fock states.

We consider a pure state that is an arbitrary superposition of n-particle states of a harmonic oscillator:

$$|\psi\rangle_\Sigma = \sum_{k=0}^{\infty} c_k |k\rangle, \qquad \sum_{k=0}^{\infty} |c_k|^2 = 1. \tag{32}$$

Its Husimi function is

$$Q_\Sigma(q,p) = \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{(\alpha^*)^k}{\sqrt{k!}} c_k e^{-|\alpha|^2/2} \sum_{s=0}^{\infty} \frac{\alpha^s}{\sqrt{s!}} c_s^*. \tag{33}$$

After a λ-scale transformation, Husimi function (33) becomes

$$Q_{\Sigma}^{\lambda}(q,p) = \lambda^2 Q(\lambda q, \lambda p) = \lambda^2 e^{-\lambda^2|\alpha|^2} \sum_{k=0}^{\infty} \frac{\lambda^k (\alpha^*)^k}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^s}{\sqrt{s!}} c_s^* =$$
$$= e^{-|\alpha|^2} \lambda^2 e^{(1-\lambda^2)|\alpha|^2} \sum_{k=0}^{\infty} \frac{\lambda^k (\alpha^*)^k}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^s}{\sqrt{s!}} c_s^*. \tag{34}$$

We expand the exponential $e^{(1-\lambda^2)|\alpha|^2}$ in a series, and expression (34) then becomes

$$\lambda^2 Q_{\Sigma}(\lambda q, \lambda p) = e^{-|\alpha|^2} \lambda^2 \sum_{j=0}^{\infty} \frac{(1-\lambda^2)^j |\alpha|^{2j}}{j!} \sum_{k=0}^{\infty} \frac{\lambda^k (\alpha^*)^k}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^s}{\sqrt{s!}} c_s^*. \tag{35}$$

It can be written as

$$\lambda^2 Q_{\Sigma}(\lambda q, \lambda p) = e^{-|\alpha|^2} \sum_{j=0}^{\infty} \lambda^2 \frac{(1-\lambda^2)^j}{j!} \sum_{k=0}^{\infty} \frac{\lambda^k (\alpha^*)^{k+j}}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^{s+j}}{\sqrt{s!}} c_s^*. \tag{36}$$

As before, we must now take relations (26) into account. Using them, we write equality (36) in the form

$$\lambda^2 Q_{\Sigma}(\lambda q, \lambda p) = \sum_{j=0}^{\infty} \lambda^2 \frac{(1-\lambda^2)^j}{j!} \sum_{k=0}^{\infty} \sqrt{\frac{(k+j)!}{k!}} \lambda^k c_k \langle \alpha | k+j \rangle \sum_{s=0}^{\infty} \sqrt{\frac{(s+j)!}{s!}} \lambda^s c_s^* \langle s+j | \alpha \rangle. \tag{37}$$

We see that expression (37) is the Husimi function of a state with the density matrix

$$\hat{\rho}_{\Sigma}^{\lambda} = \sum_{j=0}^{\infty} \frac{\lambda^2 (1-\lambda^2)^j}{j!} \left(\sum_{k=0}^{\infty} \sqrt{\frac{(k+j)!}{k!}} \lambda^k c_k | k+j \rangle \right) \left(\sum_{s=0}^{\infty} \sqrt{\frac{(s+j)!}{s!}} \lambda^s c_s^* \langle s+j | \right). \tag{38}$$

We find that after λ-scale transformation (3), pure state (32) becomes a mixed state described by density matrix (38).

We now consider the structure of density matrix (38). For this, as an example, we use density matrix (28) corresponding to the stretched state obtained from a state that is a superposition of two Fock states. For such a state, the main diagonal of the density matrix has the form of the sum of the main diagonals of the density matrices corresponding to the single Fock states, and the elements of each of these matrices are multiplied by the square of the absolute value of the coefficient with which this single state is included in the superposition.

If state (32) is a superposition of more than two n-particle states, then the main diagonal of density matrix (38) of the corresponding stretched state has a similar structure. Namely, it has the form of a sum of the main diagonals of the density matrices corresponding to the single n-particle states, and elements of each of them are multiplied by $|c_k|^2$, the square of the absolute value of the coefficient with which this single state $|k\rangle$ is included in superposition (32).

We introduce the notation

$$\begin{aligned} d_{N,j} &= \lambda^{N+1} \sqrt{\frac{(1-\lambda^2)^j (N+j)!}{N!\, j!}} c_N, \\ d_{N,j}^* &= \lambda^{N+1} \sqrt{\frac{(1-\lambda^2)^j (N+j)!}{N!\, j!}} c_N^*, \end{aligned} \qquad j = 0, 1, \ldots . \tag{39}$$

With this notation, the elements on the main diagonal of density matrix (28) become

$$D_{n,n} = \sum_{i=0}^{n} |d_{i,n-i}|^2, \quad i = 0, 1, \ldots, n, \quad n = 0, 1, \ldots. \tag{40}$$

In addition to the main diagonal, there are nonzero elements of density matrix (38) on other diagonals parallel to the main one. Their structure is analogous to the structure of the secondary diagonals of density matrix (28). We describe elements on these diagonals. For this, we use notation (31). In fact, these elements are sums of quantities of type (31).

We first consider the diagonal nearest to the main diagonal of matrix (28) and located above it. Its elements have the coordinates $(n, n+1)$, $n = 0, 1, 2, \ldots$ and are expressed as

$$D_{n,n+1} = \sum_{j=0}^{n} d_{j,n-j} d^*_{j+1,n-j}. \tag{41}$$

The summation limits in this formula are defined by the k-particle states included in superposition state (32). If state (32) contains a finite number of k-particle states and the highest of them is the state $|K\rangle$, then formula (41) becomes

$$D_{n,n+1} = \begin{cases} \displaystyle\sum_{j=0}^{n} d_{j,n-j} d^*_{j+1,n-j}, & n < K, \\ \displaystyle\sum_{j=0}^{K-1} d_{j,n-j} d^*_{j+1,n-j}, & n \geq K. \end{cases} \tag{42}$$

The values of the elements on other diagonals above the main diagonal are defined as

$$D_{n,n+k} = \begin{cases} \displaystyle\sum_{j=0}^{n} d_{j,n-j} d^*_{j+k,n-j}, & n \leq K-k, \\ \displaystyle\sum_{j=0}^{K-k} d_{j,n-j} d^*_{j+k,n-j}, & n \geq K-k. \end{cases} \tag{43}$$

If some terms in state (32) are absent, i.e., some coefficients $c_k = 0$, then the corresponding quantities $d_{k,j} = 0$, $j = 0, 1, \ldots$, and the corresponding terms in sum (39) are absent. The values of the elements of the density matrix below the main diagonal are defined by the Hermitian property:

$$D_{n+k,n} = D^*_{n,n+k}. \tag{44}$$

This is the structure of density matrix (38) of the stretched state corresponding to state (32). We study some of its properties below, but we first note the following fact. The elements on the main diagonal of matrix (38) are linear combinations of terms from negative binomial distributions (21). In our case, these distributions differ one from another by the parameter N, while the parameter λ is the same in all of them. The elements on the diagonals above and below the main diagonal are linear combinations of the distribution terms of the form

$$F(N, M; \lambda)_j = \frac{(1-\lambda^2)^j}{j!} \lambda^{N+M+2} \sqrt{\frac{(N+j)!(M+j)!}{N!\,M!}}, \quad j = 0, 1, \ldots. \tag{45}$$

Distributions (45) are characterized by the three parameters N, M, and λ, where N and M are integers and $\lambda^2 \leq 1$. For $N = M$, they become the usual negative binomial distributions. We call distributions (45) double negative binomial distributions. Their properties will be studied in another paper.

5. Means of the state number operator

We now study some properties of stretched states. First, we find the means of the particle number of stretched state (38). We do this for state (21) first.

The mean of particles in the state given by the density matrix ρ is defined by the expression

$$\langle \hat{n} \rangle = \mathrm{Tr}(\hat{N}\hat{\rho}), \tag{46}$$

where $\hat{N} = \hat{a}^+\hat{a}$ is the particle number operator.

For state (21), we have

$$\langle \hat{n}_N \rangle = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!}\left(\frac{(N+j)!}{N!}\lambda^{2N}(N+j)\right). \tag{47}$$

It is known that we have the relation

$$S_N^0(x) = \sum_{j=0}^{\infty} \frac{(N+j)!}{j!}x^j = \frac{N!}{(1-x)^{N+1}}, \quad |x| < 1. \tag{48}$$

Substituting $x = 1-\lambda^2$ in it, we obtain

$$\sum_{j=0}^{\infty} \lambda^{2N+2}\frac{(N+j)!}{N!\,j!}(1-\lambda^2)^j = 1. \tag{49}$$

We now calculate the quantity

$$S_N^1(x) = \sum_{j=0}^{\infty} j\frac{(N+j)!}{j!}x^j = x\frac{d}{dx}S_N^0(x) = \frac{x(N+1)!}{(1-x)^{N+2}}. \tag{50}$$

Using relations (49) and (50), we obtain the expression for the mean of the particle number in stretched state (21)

$$\langle \hat{n}_N \rangle = N + (S_N^0)^{-1}S_N^1 = N + \frac{(N+1)(1-\lambda^2)}{\lambda^2} = \frac{N+1}{\lambda^2} - 1. \tag{51}$$

We now find an expression for the dispersion of the particle number in state (21), which is defined as

$$\sigma_n = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{52}$$

To calculate its value, we need the mean $\langle \hat{n}_N^2 \rangle$. It can be found:

$$\begin{aligned}\langle \hat{n}_N^2 \rangle &= \sum_{j=0}^{\infty} \lambda^{2N+2}\frac{(1-\lambda^2)^j}{j!}\frac{(N+j)!}{N!}(N+j)^2 = \\ &= N^2 + 2N\langle n_N \rangle + \sum_{j=0}^{\infty} \lambda^{2N+2}\frac{(1-\lambda^2)^j}{j!}\frac{(N+j)!}{N!}j^2.\end{aligned} \tag{53}$$

We next have

$$\begin{aligned}S_N^2(x) &= \sum_{i=0}^{\infty} i^2\frac{(N+i)!}{i!}x^i = \\ &= x\frac{d}{dx}S_1 = \left(x\frac{d}{dx}\right)^2 S_0 = \frac{x(N+1)}{1-x} + \frac{x^2(N+1)(N+2)}{(1-x)^2}.\end{aligned} \tag{54}$$

Hence, using relations (51), we obtain the expression for the dispersion of the particle number in state (21):

$$\sigma_n = \langle \hat{n}_N^2 \rangle - \langle \hat{n}_N \rangle^2 = \frac{x(N+1)}{(1-x)^2} = \frac{(N+1)(1-\lambda^2)}{\lambda^4}. \tag{55}$$

We have found the quantities $\langle \hat{n}_N \rangle$ and $\langle \hat{n}_N^2 \rangle$. We now obtain $\langle \hat{n}_N^l \rangle$ for state (21). In this case, we have

$$\begin{aligned} \langle \hat{n}_N^l \rangle &= \sum_{j=0}^{\infty} \lambda^{2N+2} \frac{(1-\lambda^2)^j}{j!} \frac{(N+j)!}{N!} (N+j)^l = \\ &= \sum_{j=0}^{\infty} F_{N+j}^N \sum_{k=0}^{l} \binom{l}{k} j^k N^{l-k} = \sum_{k=0}^{l} G_N^k(\lambda) \binom{l}{k} N^{l-k}. \end{aligned} \tag{56}$$

Here, $\binom{l}{k} = \frac{l!}{(l-k)!\,k!}$ are the usual binomial coefficients, and the quantities $G_N^k(\lambda)$ are

$$G_N^k(\lambda) = \sum_{j=0}^{\infty} \lambda^{2N+2} \frac{(1-\lambda^2)^j}{j!} \frac{(N+j)!}{N!} j^k. \tag{57}$$

We calculate these sums. As shown above, equality (48) holds. Using it, we obtain

$$\begin{aligned} S_N^1(x) &= \sum_{i=0}^{\infty} i \frac{(N+i)!}{i!} x^i = x \frac{d}{dx} S_0, \\ S_N^2(x) &= \sum_{i=0}^{\infty} i^2 \frac{(N+i)!}{i!} x^i = x \frac{d}{dx} S_1 = \left(x \frac{d}{dx} \right)^2 S_0, \\ &\vdots \\ S_N^k(x) &= \sum_{i=0}^{\infty} i^k \frac{(N+i)!}{i!} x^i = \left(x \frac{d}{dx} \right)^k S_0, \\ &\vdots \end{aligned} \tag{58}$$

Using these relations, we obtain expressions for the $G_N^k(\lambda)$:

$$G_N^k(\lambda) = (S_N^0(1-\lambda^2))^{-1} S_N^k(1-\lambda^2). \tag{59}$$

Hence, we have the formula

$$\langle \hat{n}^l \rangle = \sum_{k=0}^{l} (S_N^0(1-\lambda^2))^{-1} S_N^k(1-\lambda^2) \binom{l}{k} N^{l-k}. \tag{60}$$

We note that both positive and negative binomial coefficients are used to construct the $\langle \hat{n}_N^l \rangle$.

We now calculate the values of quantities (51) and (60) for the stretched states with density matrix (38). We first do this for state (28). In this case, the mean particle number $\langle \hat{n}_{N,V} \rangle$ is

$$\begin{aligned} \langle \hat{n}_{N,V} \rangle &= \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!} \left(\frac{(N+j)!}{N!} \lambda^{2N} |c_N|^2 (N+j) + \frac{(M+j)!}{M!} \lambda^{2M} |c_M|^2 (M+j) \right) = \\ &= \frac{1}{\lambda^2} \left(|c_N|^2 N + |c_M|^2 M \right) + \frac{1}{\lambda^2} - 1, \qquad |c_N|^2 + |c_M|^2 = 1. \end{aligned} \tag{61}$$

The mean of an arbitrary power of the particle number operator $\langle(\hat{n}_{N,V})^l\rangle$ for state (28) is

$$\langle(\hat{n}_{N,V})^l\rangle = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!}\left(\frac{(N+j)!}{N!}\lambda^{2N}|c_N|^2(N+j)^l + \frac{(M+j)!}{M!}\lambda^{2M}|c_M|^2(M+j)^l\right) =$$

$$= \frac{1}{\lambda^2}\big(|c_N|^2\langle\hat{n}_N^l\rangle + |c_M|^2\langle\hat{n}_M^l\rangle\big) + \frac{1}{\lambda^2} - 1, \qquad |c_N|^2 + |c_M|^2 = 1. \tag{62}$$

In the case of state (38) of general form, the quantities $\langle\hat{n}_\Sigma\rangle$ and $\langle(\hat{n}_\Sigma)^l\rangle$ can be written as

$$\begin{aligned} \langle\hat{n}_\Sigma\rangle &= \frac{1}{\lambda^2}\sum_{k=0}^{\infty}|c_k|^2 k + \frac{1}{\lambda^2} - 1, \\ \langle(\hat{n}_\Sigma)^l\rangle &= \frac{1}{\lambda^2}\sum_{k=0}^{\infty}|c_k|^2\langle\hat{n}_k^l\rangle + \frac{1}{\lambda^2} - 1, \end{aligned} \qquad \sum_{k=0}^{\infty}|c_k|^2 = 1. \tag{63}$$

We see that formulas (63) for the means of powers of the particle number operator for state (38) are the sums of similar expressions for state (21). This can be explained because only the diagonal elements of density matrix (38) are included in formulas (63) and these elements equal the sum of diagonal elements of density matrices (21) for the stretched states $|k\rangle$ multiplied by the squares of the absolute values of the coefficients c_k with which they are included in state (32).

6. Entropy of stretched states

We now consider the von Neumann entropy H of stretched states. Let there be a state defined by a density matrix $\hat{\rho}$. Then the quantity H is

$$H = -\sum_k \lambda_k \log_2 \lambda_k. \tag{64}$$

Here, λ_k is an eigenvalue of the density matrix ρ. Moreover, we assume that $0 \cdot \log_2 0 = 0$. It follows from this definition that the entropy of a pure state is equal to zero because the idempotency of the density matrix of the pure state $\hat{\rho}^2 = \hat{\rho}$ implies that some of its eigenvalues are unity and the others are zero.

If we have a mixed state, then its entropy is defined by the distribution of the λ_k. The value λ_k defines the probability of finding the system in the state associated with the given eigenvalue of the density matrix. It is easy to understand that the more uniformly the probabilities λ_k are distributed, the greater entropy (64) is, and it reaches its maximum when these probabilities are equal to each other. In the case where sum (64) has N terms, the maximum of the von Neumann entropy H is at $\lambda_k = 1/N$ and is $H = \log_2 N$.

We now consider the stretched states corresponding to the N-particle state of the harmonic oscillator. Its density matrix is

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|, \quad \lambda^2 < 1. \tag{65}$$

Because this matrix is diagonal, its elements

$$c_k^N = \frac{\lambda^{2N+2}(N+k)!}{N!\,k!}(1-\lambda^2)^k \tag{66}$$

are its eigenvalues, and they can be directly substituted in expression (64).

Therefore, to find the von Neumann entropy of state (65), we must calculate the quantity

$$H_N = -\sum_{k=0}^{\infty} c_k^N \log_2 c_k^N = \\ = -\sum_{k=0}^{\infty} \frac{\lambda^{2N+2}(N+k)!}{N!\,k!}(1-\lambda^2)^k \log_2\biggl(\frac{\lambda^{2N+2}(N+k)!}{N!\,k!}(1-\lambda^2)^k\biggr). \tag{67}$$

If $N = 0$, then we have

$$H_0 = -\lambda^2 \sum_{k=0}^{\infty}(1-\lambda^2)^k \log_2(\lambda^2(1-\lambda^2)^k) = -\log_2 \lambda^2 - \log_2(1-\lambda^2)\frac{1-\lambda^2}{\lambda^2}. \tag{68}$$

If $\lambda \to 1$, then $H_0 \to 0$, and this corresponds to the fact that the entropy of a pure N-particle state is zero. If $\lambda \to 0$, then $H_0 \to \infty$. This means that the uncertainty in the probability of detecting an arbitrary n-particle state increases.

If there is a stretched state corresponding to superposition (32) of n-particle states of a harmonic oscillator, then its density matrix (38) is not diagonal, and we must find its eigenvalues to calculate the entropy of such a state.

7. Uncertainty relation for stretched states

In the preceding section, we calculated the means of the particle number operator and its powers for the stretched state obtained from an arbitrary superposition of k-particle Fock states of a harmonic oscillator. In this section, we consider some other operators and present the form of the Heisenberg and Robertson–Schrödinger uncertainty relations for stretched states. In [22], a new method for constructing the Husimi symbols was proposed. It is especially effective for an operator that is polynomial in the coordinate and momentum operators $\hat{q}$ and $\hat{p}$. These are the operators we treat here.

We first consider the harmonic oscillator Hamiltonian

$$\widehat{H} = \frac{\hbar\omega}{2}\bigl(\hat{q}^2 + \hat{p}^2\bigr). \tag{69}$$

Its Husimi symbol is

$$K_{\mathrm{H}}(q,p) = q^2 + p^2 - 1. \tag{70}$$

The mean energy $\bar{E}$ of the state characterized by the Husimi function $Q(q,p)$ is

$$\bar{E} = \int \frac{\hbar\omega}{2}(q^2 + p^2 - 1)Q(q,p)\,dq\,dp = \int \frac{\hbar\omega}{2}(q^2+p^2)Q(q,p)\,dq\,dp - \frac{\hbar\omega}{2}. \tag{71}$$

We now find the mean energy of the stretched state corresponding to the state with the Husimi function $Q(q,p)$. The Husimi function of the stretched state can be written as $Q_\lambda(q,p) = \lambda^2 Q(\lambda q, \lambda p)$, and the mean energy $\bar{E}_\lambda$ of the state with such a Husimi function is defined by

$$\bar{E}_\lambda = \int K_{\mathrm{H}}(q,p) Q_\lambda(q,p)\,dq\,dp = \\ = \int \frac{1}{\lambda^2}\frac{\hbar\omega}{2}\bigl((\lambda q)^2 + (\lambda p)^2 - 1\bigr)Q(\lambda q, \lambda p)\,d(\lambda q)\,d(\lambda p) = \frac{1}{\lambda^2}\bar{E} + \frac{1-\lambda^2}{\lambda^2}\frac{\hbar\omega}{2}. \tag{72}$$

This expression holds for all stretched states of a harmonic oscillator. Because $|\lambda|^2 < 1$, we can recognize that the energy of a stretched state increases under a scale transformation $(q,p) \to (\lambda q, \lambda p)$. For the Fock states, the mean energy of the corresponding stretched states can be found explicitly. Hence, in the case of superposition (32), we have

$$\bar{E}_{\Sigma\lambda} = \frac{1}{\lambda^2}\frac{\hbar\omega}{2}\sum_{k=0}^{\infty}|c_k|^2 k + \frac{1-\lambda^2}{\lambda^2}\frac{\hbar\omega}{2}. \tag{73}$$

Using the expression for the means of operators in terms of the Husimi functions of states, we can establish the form of the uncertainty relations for the stretched states. We consider the Heisenberg and Robertson–Schrödinger uncertainty relations here. They can be written in the general forms

$$\sigma_{qq}\sigma_{pp} \geq \frac{1}{4}\hbar^2, \qquad \sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq \frac{1}{4}\hbar^2, \tag{74}$$

and they hold for any quantum state. We now determine what happens with these relations when we pass to the stretched states. For this, we must calculate the quantities

$$\sigma_{qq} = \langle \hat{q}^2\rangle - \langle \hat{q}\rangle^2, \qquad \sigma_{pp} = \langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2, \qquad \sigma_{qp} = \frac{1}{2}\langle \hat{p}\hat{q} + \hat{q}\hat{p}\rangle - \langle \hat{q}\rangle\langle \hat{p}\rangle.$$

Using the Husimi function, we can write the dispersions σ_{qq} and σ_{pp} and the quantity σ_{qp} as

$$\begin{aligned}
\sigma_{qq} &= \int\left(q^2 - \frac{1}{2}\right)Q(q,p)\,dq\,dp - \left(\int qQ(q,p)\,dq\,dp\right)^2,\\
\sigma_{pp} &= \int\left(p^2 - \frac{1}{2}\right)Q(q,p)\,dq\,dp - \left(\int pQ(q,p)\,dq\,dp\right)^2,\\
\sigma_{qp} &= \int qpQ(q,p)\,dq\,dp - \int qQ(q,p)\,dq\,dp\int qQ(q,p)\,dq\,dp.
\end{aligned} \tag{75}$$

For stretched states. these formulas become

$$\begin{aligned}
\sigma_{qq\lambda} &= \int\left(q^2 - \frac{1}{2}\right)\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp - \left(\int q\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp\right)^2,\\
\sigma_{pp\lambda} &= \int\left(p^2 - \frac{1}{2}\right)\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp - \left(\int p\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp\right)^2,\\
\sigma_{qp\lambda} &= \int qp\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp - \int q\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp\int q\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp.
\end{aligned} \tag{76}$$

These expressions yield the values of σ_{qq}, σ_{pp}, and σ_{qp} for stretched states:

$$\sigma_{qq\lambda} = \frac{1}{\lambda^2}\sigma_{qq} + \frac{1-\lambda^2}{\lambda^2}, \qquad \sigma_{pp\lambda} = \frac{1}{\lambda^2}\sigma_{pp} + \frac{1-\lambda^2}{\lambda^2}, \qquad \sigma_{qp\lambda} = \frac{1}{\lambda^2}\sigma_{qp}. \tag{77}$$

Using the obtained expressions, we find the modification of the Heisenberg and Robertson–Schrödinger uncertainty relations in passing to the stretched states,

$$\begin{aligned}
\sigma_{qq\lambda}\sigma_{pp\lambda} &= \frac{1}{\lambda^4}\left(\sigma_{qq}\sigma_{pp} + \frac{1}{2}(1-\lambda^2)(\sigma_{qq}+\sigma_{pp}) + \frac{1}{4}(1-\lambda^2)^2\right) \geq \frac{1}{4\lambda^4}\hbar^2,\\
\sigma_{qq\lambda}\sigma_{pp\lambda} - \sigma_{qp\lambda}^2 &= \frac{1}{\lambda^4}\left(\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 + \frac{1}{2}(1-\lambda^2)(\sigma_{qq}+\sigma_{pp}) + \frac{1}{4}(1-\lambda^2)^2\right) \geq \frac{1}{4\lambda^4}\hbar^2.
\end{aligned}$$

We see that the right-hand sides of these uncertainty relations contain the factor λ^{-4}. For $|\lambda| < 1$, their values therefore increase and, in general, can become arbitrarily large. Some other states have the same property, for instance, the so-called correlated states that arise as a generalization of coherent states [23]. These states have a large coordinate and momentum uncertainty. In [24]–[26], it was proposed to use this property of such states to describe some phenomena in which the observed probability of transmission through the potential barrier is larger than usual. The increase in the tunneling probability can be formally associated with an "increase" of the Planck constant $\hbar$, i.e., we can assume that the scale transformation $(q,p) \to (\lambda q, \lambda p)$ generates an "effective Planck constant" $\hbar_{\text{eff}} = \hbar/\lambda^2$. The effective Planck constant for $\lambda^2 \ll 1$ satisfies the inequality $\hbar_{\text{eff}} \gg \hbar$.

8. The case of large $\boldsymbol{\lambda}$

Up to now in our considerations, we assumed that $\lambda < 1$. Moreover, we relied on a theorem proved in [17]. According to this theorem, if $Q(q,p)$ is the Husimi function of a quantum state and $\lambda < 1$, then $\lambda^2 Q(\lambda q, \lambda p)$ is also a Husimi function of some quantum state. There is no known statements of this type for $\lambda > 1$. To clarify what kind of problems can arise in this case, we try to implement the construction used for $\lambda < 1$ to the case $\lambda > 1$.

We first consider the case $1 < \lambda < 2$ and choose the N-particle Fock state $|N\rangle$. Its Husimi function is

$$Q^N(q,p) = e^{-|\alpha|^2} \frac{|\alpha|^{2N}}{N!}. \tag{78}$$

After the transformation $(q,p) \to (\lambda q, \lambda p)$, it becomes

$$\lambda^2 Q^N(\lambda q, \lambda p) = \lambda^{2+2N} e^{-\lambda^2|\alpha|^2} \frac{|\alpha|^{2N}}{N!} = e^{-4|\alpha|^2} \lambda^{2+2N} e^{(4-\lambda^2)|\alpha|^2} \frac{|\alpha|^{2N}}{N!}. \tag{79}$$

As before, we want to represent this function as a sum of quantities of form (78). For this, we expand the exponential $e^{(4-\lambda^2)|\alpha|^2}$ in a series:

$$e^{(4-\lambda^2)|\alpha|^2} = \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j}{j!} |\alpha|^{2j}. \tag{80}$$

Substituting this expansion in (79), for $1 < \lambda < 2$, we obtain

$$\begin{aligned}
\lambda^2 Q^N(\lambda q, \lambda p) &= \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j}{N!\,j!} \lambda^{2+2N} e^{-4|\alpha|^2} |\alpha|^{2N+2j} = \\
&= \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j}{N!\,j!} \lambda^{2+2N} e^{-4|\alpha|^2} \frac{(4|\alpha|^2)^{N+j}}{4^{N+j}} = \\
&= \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j (N+j)!}{N!\,j!} \lambda^{2+2N} \frac{1}{4^{N+j}} \left(e^{-4|\alpha|^2} \frac{(4|\alpha|^2)^{N+j}}{(N+j)!} \right) = \\
&= 4 \sum_{j=0}^{\infty} \frac{(N+j)!}{N!\,j!} \left(1 - \frac{\lambda^2}{4}\right)^j \left(\frac{\lambda^2}{4}\right)^{1+N} Q^{N+j}(2q, 2p).
\end{aligned} \tag{81}$$

In the general case where $s-1 < \lambda < s$, we have

$$\lambda^2 Q^N(\lambda q, \lambda p) = s^2 \sum_{j=0}^{\infty} \frac{(N+j)!}{N!\,j!} \left(1 - \frac{\lambda^2}{s^2}\right)^j \left(\frac{\lambda^2}{s^2}\right)^{1+N} Q^{N+j}(sq, sp). \tag{82}$$

As before, the coefficients of the quantities $Q^{N+j}(sq, sp)$ in sums (81) and (82) are the elements of a negative binomial distribution. Their sum is equal to unity, and series (81) and (82) therefore converge. Hence, the only open problem is to interpret the quantities

$$Q^N(sq, sp) = s^{2N} e^{-s^2|\alpha|^2} \frac{|\alpha|^{2N}}{N!} = e^{-s^2q^2 - s^2p^2} \frac{(s^2q^2 + s^2p^2)^N}{N!}. \tag{83}$$

In [27], the problem of the conditions that must be satisfied by a function $F(q, p)$ (defined on the phase space) for this function to be the Husimi function of a quantum state was studied. It was shown that the Gauss function

$$F(q, p) = Ce^{-\lambda^2q^2 - \lambda^2p^2} \tag{84}$$

is the Husimi function of a quantum state if and only if $\lambda \leq 1$. Functions (83) do not satisfy this criterion; consequently, no quantum state is associated with them, i.e., they are not Husimi functions. Therefore, scale transformation (3) for $\lambda^2 > 1$, in general, does not transform a Husimi function into another Husimi function.

9. Conclusion

We have proposed a new method for constructing the states arising as a result of quantum processes. This method is based on using quasiprobability distributions, more specifically, Husimi functions. The chain of arguments is as follows. Let some process be described by a Hamiltonian H. Using this Hamiltonian, we can find an operator equation for the density matrix ρ. Using the relation between the Husimi function Q and the density matrix ρ, we can pass from this operator equation to an ordinary differential equation for the Husimi function.

The principle underlying our method is that solutions of a given differential equation are associated with transformations of a phase space. Such a transformation induces a transformation of the Husimi functions defined in it. Expressing such transformed functions $\widetilde{Q}$ in terms of the already known Husimi functions, we can perform the inverse transformation and find the density matrix $\tilde{\rho}$ of the transformed state. Here, we implemented this program for scale transformation (3) in the phase space, which models the operation of a quantum amplifier [18], [19]. We constructed the density matrix for a stretched state arising as a result of the action of this amplifier on n-particle states of a harmonic oscillator and on their superpositions. We plan to consider other states and other transformations of the phase space in the future, in particular, states leading to weakening of Fock states rather than their amplification.

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