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Derivation of the Husimi symbols without antinormal ordering, scale transformation and uncertainty relations

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Abstract

We propose a new method for the derivation of Husimi symbols, for operators that are given in the form of products of an arbitrary number of coordinates, and momentum operators, in an arbitrary order. For such an operator, in the standard approach, one expresses coordinate and momentum operators as a linear combination of the creation and annihilation operators, and then uses the antinormal ordering to obtain the final form of the symbol. In our method, one obtains the Husimi symbol in a much more straightforward fashion, departing directly from operator explicit form without transforming it through creation and annihilation operators. With this method the mean values of some operators are found. It is shown how the Heisenberg and the Schrödinger–Robertson uncertainty relations, for position and momentum, are transformed under scale transformation $(q; p) \to (\lambda q; \lambda p)$. The physical sense of some states which can be constructed with this transformation is also discussed.

Keywords: average values, Husimi function, polynomials, operators, scaling transform, uncertainty relation

1. Introduction

In classical statistical mechanics, in order to find the mean value of any function $F(q, p)$ defined on the phase space, one has to integrate that function over the phase space, weighted with an appropriate probability density function, i.e.

$$\langle F \rangle = \int F(q, p)\rho(q, p)\mathrm{d}q\mathrm{d}p. \tag{1}$$

Here, $\rho(q, p)$ is a probability density, which means that the integral of this function over a certain region of phase space gives the probability of having the system in that region of the phase space. In quantum mechanics, to each observable $F(q, p)$ one assigns an operator $\hat{A}$. However, if one wants to keep the resemblance to classical statistical mechanics, i.e. to still compute the mean value of an operator by some formula similar to (1), some additional steps are needed. First, an analogue of the classical distribution function has to be chosen, and this analogue comes in the form of the quasidistribution function $D(q, p)$. The mean value is now calculated similarly to (1) as:

$$\langle \hat{A} \rangle = \int A_D(q, p)D_\rho(q, p)\mathrm{d}q\mathrm{d}p. \tag{2}$$

A few notes about the function $A_D(q, p)$ are in order. This function should be assigned to each operator $\hat{A}$. The process of assignment is far from trivial, and is one the main themes of this work. The function is defined on the whole phase space, and must fulfil the condition that the mean value $\langle \hat{A} \rangle$ of an operator $\hat{A}$ at a given state, described by the

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quasidistribution $D(q, p)$, is given by (2). The function $A_D(q, p)$ is called the symbol of the operator $\hat{A}$, corresponding to the quasidistribution $D(q, p)$. In quantum mechanics a number of quasidistribution functions are used, hence, different symbols are assigned to the same operator, $\hat{A}$, depending on the quasidistribution [1–5]. So, a problem arises —how do we find the symbol of the given operator for the concrete quasidistribution?

In a number of cases the answer to this question is known. Concretely, the symbols are known for the following quasidistributions: Wigner function $W(q, p)$ [8], Husimi-Kano $Q(q, p)$ [6, 7] and Glauber-Sudarsan $P(q, p)$ [9, 10]. If the operator $\hat{A}$ comes in the form of creation and annihilation operators' bivariate polynomial, then its W, Q, and P-symbols are obtained using operations of symmetrization, antinormal and normal ordering [11–13].

The sequence of the actions is as follows: first, in a given operator which is the function of the coordinate and momentum operators, one expresses the mentioned operators as corresponding linear combinations of creation and annihilation operators. To this form of the operator the procedure of symmetrization or antisymmetrization, depending on the quasiprobability used, is applied. For this ordered form the scalar function can be directly obtained. This procedure, in principle, solves the problem of the determination of the symbols for the polynomial operators, but if the polynomial is complicated, the procedure can be very tedious.

In this paper, we propose a new, simpler, way of determining the Q-symbols. In our approach, we simply replace operators $\hat{q}$ and $\hat{p}$ at the places where they stand in the original operator, with differential operators which we will define below. This is done explicitly, without using any operations such as symmetrization or operator ordering. The obtained differential operator acts on the Husimi function giving the appropriate Q- symbol of the original operator. The crucial fact is that for obtaining the Q- symbol of an operator, the explicit form of the Husimi function for a given state is not needed. One uses only its general structure which is the same for all concrete Husimi functions. In section 2 we give the Husimi function in a form that is used below. Our novel method is presented in section 3.

In section 4, with the help of this method, we evaluate the mean values of some operators and analyze the behavior of the uncertainty relations under scale transformation $(q; p) \to (\lambda q; \lambda p)$. We show that as a result of the transformation, the right-hand side of the inequalities increases. This result can be used to explain some of the tunnelling phenomena.

In section 5 we investigate the properties of 'stretched Fock states'. These states can be achieved by applying of the scale transformation to Fock states of the harmonic oscillator.

2. Husimi function and the mean value problem

The Husimi functions are determined by the density operator and the set of coherent states of a harmonic oscillator [6, 7]. Let us consider some state, described by the density operator $\hat{\rho}$, and $\langle x|\alpha\rangle$ is a coherent state. Then, the Husimi function of the state $\hat{\rho}$ is defined by

$$Q\left(\alpha, \alpha^*\right) = \frac{1}{\pi}\int \langle \alpha \mid x\rangle \rho(x, y)\langle y \mid \alpha\rangle \mathrm{d}x\mathrm{d}y. \tag{3}$$

Here, $\alpha = \alpha_r + \mathrm{i}\alpha_i$ is an arbitrary complex number, and $\rho(x, y)$ is the kernel of the density operator in the coordinate representation. For the complex number α, which determines the coherent state, we will use expression $\alpha = (q + \mathrm{i}p)/\sqrt{2}$ and will regard the Husimi function Q as a function of p and q

$$Q(q, p) = \frac{1}{2\pi\hbar}\langle q, p\,|\hat{\rho}|\,q, p\rangle. \tag{4}$$

Here $|q, p\rangle$ is the coherent state, given in terms of the variables q and p:

$$\langle x \mid q, p\rangle = \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}(x-q)^2 + \mathrm{i}px - \frac{\mathrm{i}}{2}qp\right]. \tag{5}$$

The Husimi function of the state, given by the wave function $\psi(x)$, has the following form

$$\begin{aligned} Q(q, p) &= \frac{1}{2}\left(\frac{1}{\pi}\right)^{3/2}\int \exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right]\psi^*(y) \\ &\quad\times \psi(x)\exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right]\mathrm{d}x\mathrm{d}y \\ &= C\int F(q, p; x. y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{6}$$

Using the Husimi function one can also determine the mean value of the operator, by applying the formula (2).

The Q-symbol of the operator, $\hat{A}$, in the standard approach, is determined with the help of the antinormal ordering of the creation and annihilation operators in the expression for the operator, $\hat{A}$. If the operator, $\hat{A}$, is a low order polynomial of the creation and annihilation operators, or of coordinates and momenta, then one can find the explicit form of its symbols by performing the antinormal ordering. In a number of cases, however, the procedure could be tedious. In this paper we propose a new procedure for determining the Q- symbols of an operator of arbitrary form, without using any operator ordering.

3. Deriving the Husimi symbols without antinormal ordering

Let us introduce the operator

$$\hat{X} = q + \frac{1}{2}\frac{\partial}{\partial q} + \frac{\mathrm{i}}{2}\frac{\partial}{\partial p}. \tag{7}$$

Using the explicit form of the function (6), one can prove the following formula

$$\begin{aligned} \int \hat{X}Q(q, p)\mathrm{d}q\mathrm{d}p &= \int qQ(q, p)\mathrm{d}q\mathrm{d}p \\ &= \int \psi(x)x\psi^*(x)\mathrm{d}x. \end{aligned} \tag{8}$$

The right-hand side of (8) is, by definition, the mean value of the coordinate operator. So, this equation shows that the mean value of the coordinate operator, when the state is described by the Husimi function, may be represented by the left-hand side of (8). So, in order to find the mean value of the coordinate operator $\hat{x}$, when the quantum state is described by the Husimi function, one should apply the operator $\hat{X}$ (7) on the Husimi function of the corresponding state and integrate the obtained result over the whole phase space (q, p). The equation (8) presents two alternative expressions for the mean value of the coordinate operator, one using the wave function for the description of the states and the other using the Husimi function.

Let us generalize the obtained results to the case of arbitrary exponent of the coordinate operator. It can be easily seen, that the mean value of some operator $K(\hat{X})$, where K is a polynomial of one variable, can be calculated by the following formula

$$\left\langle K\left(\hat{X}\right)\right\rangle = C\int \tilde{K}(q)F(q, p; x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}q\mathrm{d}p. \tag{9}$$

In the case where $K(\hat{X})$ is a monomial $K(\hat{X}) = (\hat{X})^n$ and with the help of mathematical induction one can prove that the functions $\tilde{K}_n(q)$ are related by the recurrent relation

$$\tilde{K}_{n+1}(q) = q\tilde{K}_n(q) - \frac{1}{2}\frac{\partial}{\partial q}\tilde{K}_n(q), \quad \tilde{K}_0 = 1. \tag{10}$$

$$\tilde{K}_n(q) = \sum_{0\leqslant s\leqslant[n/2]}\frac{(-1)^s n!}{s!(n-2s)!4^s}q^{n-2s}. \tag{11}$$

Here $[n/2]$ is the integer part of a number $n/2$.

So, in order to derive the mean value of the operator $K(\hat{X})$, one has to apply this operator to the Husimi function and integrate over the parameter space (q, p), which determines the coherent state (5). The result is a polynomial of the variable q.

Almost the same procedure can be used in order to obtain the mean values of the momenta operators. To this end, in addition to the operator $\hat{X}$ defined in (7), consider the operator $\hat{P}$ defined as:

$$\hat{P} = p - \frac{\mathrm{i}}{2}\frac{\partial}{\partial q} + \frac{1}{2}\frac{\partial}{\partial p}. \tag{12}$$

Operators (7), (12) satisfy the commutation relation

$$[\hat{P}, \hat{X}] = -i. \tag{13}$$

For the operator $\hat{P}$ the following relation holds

$$\begin{aligned}\hat{P}Q &= C\int\left(p - \frac{\mathrm{i}}{2}\frac{\partial}{\partial q} + \frac{1}{2}\frac{\partial}{\partial p}\right)F(q, p; x, y)\mathrm{d}x\mathrm{d}y \\ &= C\int(p + \mathrm{i}q - \mathrm{i}x)F(q, p; x, y)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{14}$$

Integrating this expression over the phase space parameters (q, p), we obtain

$$\int\hat{P}Q(q, p)\mathrm{d}q\mathrm{d}p = C\int pF(q, p; x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}q\mathrm{d}p. \tag{15}$$

By induction one can prove that

$$\begin{aligned}\left\langle\hat{p}^n\right\rangle &= \int\hat{P}^n Q(q, p)\mathrm{d}q\mathrm{d}p \\ &= C\int L_n(p)F(q, p; x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}q\mathrm{d}p.\end{aligned} \tag{16}$$

The polynomials $L_n(p)$ are related by the following recurrence relation, which is the analogue of (10)

$$\begin{aligned}L_{n+1}(p) &= pL_n(p) - \frac{1}{2}\frac{\partial}{\partial p}L_n(p), \quad L_0 = 1; \\ L_n(p) &= \sum_{0\leqslant s\leqslant[n/2]}\frac{(-1)^s n!}{s!(n-2s)!4^s}p^{n-2s}.\end{aligned} \tag{17}$$

Here $[n/2]$ is the integer part of a number $n/2$.

Let us now consider the following more general problem. Consider the operator which is the monomial of the coordinate and momenta operators. In the coordinate representation it can be represented as the polynomial of operators x and $\frac{\partial}{\partial x}$. More precisely, it will be the monomial expression

$$\hat{A}\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right) = \left(-\mathrm{i}\frac{\partial}{\partial x}\right)x\ldots x\left(-\mathrm{i}\frac{\partial}{\partial x}\right). \tag{18}$$

The mean value of this operator in the state $\psi(x)$ is determined by the formula

$$\begin{aligned}\langle\hat{A}\rangle = C\int&\psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right] \\ &\cdot\exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right] \\ &\times A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q.\end{aligned} \tag{19}$$

One can find the expression for this mean value with the help of mathematical induction. Supposing that the mean value of the operator (19) is determined by

$$\left\langle\hat{A}\right\rangle = \int\hat{P}\hat{X}\ldots\hat{X}\hat{P}Q(q, p)\mathrm{d}q\mathrm{d}p, \tag{20}$$

i.e. supposing that the following equality holds

$$\begin{aligned}&\int\hat{P}\hat{X}\ldots\hat{X}\hat{P}Q(q, p)\mathrm{d}q\mathrm{d}p \\ &= C\int\psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right] \\ &\cdot\exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right] \\ &\times A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q.\end{aligned} \tag{21}$$

Let us now consider the following operator

$$x\hat{A}\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right) = x\left(-\mathrm{i}\frac{\partial}{\partial x}\right)x\ldots x\left(-\mathrm{i}\frac{\partial}{\partial x}\right). \tag{22}$$

Its mean value has a form

$$\left\langle x\hat{A}\right\rangle = C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right] \cdot \exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right] \times xA\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q$$
$$= C\int \mathrm{d}p\mathrm{d}q\hat{X}\int \mathrm{d}x\mathrm{d}y\psi^*(x) \times \exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right] \times \exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right] \cdot A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)$$
$$= \int \hat{X}\hat{P}\hat{X}\ \ldots\ \hat{X}\hat{P}Q(q, p)\mathrm{d}p\mathrm{d}q. \tag{23}$$

This result shows that in order to obtain the mean value of the operator $x\hat{A}$, one has to apply the operator $\hat{X}\hat{P}\hat{X}\ \ldots\ \hat{X}\hat{P}$ to the Husimi function $Q(q, p)$ and integrate the obtained result over dpdq.

Analogously, one can consider the operator

$$\left(-i\frac{\partial}{\partial x}\right)\hat{A}\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right) = \left(-\mathrm{i}\frac{\partial}{\partial x}\right)\left(-\mathrm{i}\frac{\partial}{\partial x}\right)x\ \ldots\ x\left(-\mathrm{i}\frac{\partial}{\partial x}\right). \tag{24}$$

Its mean value has the form

$$\left\langle\left(-\mathrm{i}\frac{\partial}{\partial x}\right)\hat{A}\right\rangle = C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right] \cdot \exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right] \times \left(-\mathrm{i}\frac{\partial}{\partial x}\right)A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q. \tag{25}$$

Integrating by parts one obtains

$$C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py - \frac{1}{2}(x-q)^2 - \mathrm{i}px\right] \cdot (p - \mathrm{i}x + \mathrm{i}q)A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q$$
$$= C\int \mathrm{d}p\mathrm{d}q\hat{P}\int \mathrm{d}x\mathrm{d}y\psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right] \times \exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right]$$
$$A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x) = \int \hat{P}\hat{P}\hat{X}\ \ldots\ \hat{X}\hat{P}Q(q, p)\mathrm{d}p\mathrm{d}q. \tag{26}$$

From formulas (23), (26) it can be seen that, in order to determine the mean value of the operator (18), one has to apply the operators $\hat{X}$, $\hat{P}$ to the Husimi function $Q(q, p)$ in the order in which the operators x, $-\mathrm{i}\frac{\partial}{\partial x}$ appear in the operator $A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)$, and integrate this result over (q, p).

4. Evaluation of mean values and uncertainty relations

Let us consider a state with a Husimi function $Q(q, p)$. It was shown in [14] that if Q(q, p) is a Husimi function of a quantum state and $\lambda^2 < 1$, then the transformed function $\lambda^2 Q(\lambda q, \lambda p)$ is a Husimi function of some quantum state too. A state with the Husimi function $\lambda^2 Q(\lambda q, \lambda p)$ we call a stretched state.

In this section we consider the problem of evaluation of average values of some operators. Let us consider a Hamiltonian operator

$$\bar{H} = \frac{\hbar\omega}{2}\left(\hat{q}^2 + \hat{p}^2\right). \tag{27}$$

The average value of the energy $\bar{E}$ of a state with a Husimi function $Q(q, p)$ reads

$$\bar{E} = \int \frac{\hbar\omega}{2}\left(q^2 + p^2 - 1\right)Q(q, p)\mathrm{d}q\mathrm{d}p = \int \frac{\hbar\omega}{2}\left(q^2 + p^2\right)Q(q, p)\mathrm{d}q\mathrm{d}p - \frac{\hbar\omega}{2}. \tag{28}$$

The average value of the energy $\bar{E}_\lambda$ of a stretched state with a Husimi function $\lambda^2 Q(\lambda q, \lambda p)$ reads

$$\bar{E}_\lambda = \int \frac{\hbar\omega}{2}\left(q^2 + p^2 - 1\right)\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p$$
$$= \int \frac{1}{\lambda^2}\frac{\hbar\omega}{2}\left((\lambda q)^2 + (\lambda p)^2\right)Q(\lambda q, \lambda p)\mathrm{d}(\lambda q)\mathrm{d}(\lambda p) - \frac{\hbar\omega}{2}$$
$$= \frac{1}{\lambda^2}\bar{E} + \frac{1-\lambda^2}{\lambda^2}\frac{\hbar\omega}{2}. \tag{29}$$

One can see from the expression (29) that energy of a state increases after the transform $(q; p) \to (\lambda q; \lambda p)$.

Let us consider now the Heisenberg uncertainty relation

$$\sigma_{qq}\sigma_{pp} \geqslant \frac{1}{4}\hbar^2,$$
$$\sigma_{qq} = \left\langle\hat{q}^2\right\rangle - \left\langle\hat{q}\right\rangle^2, \quad \sigma_{pp} = \left\langle\hat{p}^2\right\rangle - \left\langle\hat{p}\right\rangle^2, \tag{30}$$

and the Schrödinger–Robertson uncertainty relation

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geqslant \frac{1}{4}\hbar^2;$$
$$\sigma_{qp} = \frac{1}{2}\left\langle\hat{p}\hat{q} + \hat{q}\hat{p}\right\rangle - \left\langle\hat{q}\right\rangle\left\langle\hat{p}\right\rangle. \tag{31}$$

The dispersions σ_{qq} and σ_{pp} can be evaluated with the help of the Husimi functions

$$\sigma_{qq} = \int\left(q^2 - \frac{1}{2}\right)Q(q, p)\mathrm{d}q\mathrm{d}p - \left(\int qQ(q, p)\mathrm{d}q\mathrm{d}p\right)^2,$$
$$\sigma_{pp} = \int\left(p^2 - \frac{1}{2}\right)Q(q, p)\mathrm{d}q\mathrm{d}p - \left(\int pQ(q, p)\mathrm{d}q\mathrm{d}p\right)^2. \tag{32}$$

Also

$$\sigma_{qp} = \int qpQ(q, p)\mathrm{d}q\mathrm{d}p - \int qQ(q, p)\mathrm{d}q\mathrm{d}p \int pQ(q, p)\mathrm{d}q\mathrm{d}p. \quad (33)$$

For the stretched states the formulas (32), (33) take the form

$$\begin{aligned}\sigma_{qq\lambda} &= \int \left(q^2 - \frac{1}{2}\right)\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p \\ &- \left(\int q\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p\right)^2, \\ \sigma_{pp\lambda} &= \int \left(p^2 - \frac{1}{2}\right)\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p \\ &- \left(\int p\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p\right)^2, \\ \sigma_{qp\lambda} &= \int qp\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p \\ &- \int q\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p \int p\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p. \end{aligned} \quad (34)$$

From the formulas (34) one can find values of dispersions σ_{qq} and σ_{pp} for stretched states.

$$\sigma_{qq\lambda} = \frac{1}{\lambda^2}\sigma_{qq} + \frac{1 - \lambda^2}{2\lambda^2}, \quad \sigma_{pp\lambda} = \frac{1}{\lambda^2}\sigma_{pp} + \frac{1 - \lambda^2}{2\lambda^2}, \quad \sigma_{qp\lambda} = \frac{1}{\lambda^2}\sigma_{qp}. \quad (35)$$

We see that for stretched states the Heisenberg uncertainty relation reads

$$\sigma_{qq\lambda}\sigma_{pp\lambda} = \frac{1}{\lambda^4}\left(\sigma_{qq}\sigma_{pp} + \frac{1}{2}\left(1 - \lambda^2\right)\left(\sigma_{qq} + \sigma_{pp}\right) + \frac{1}{4}\left(1 - \lambda^2\right)^2\right) \geqslant \frac{1}{4\lambda^4}\hbar^2, \quad (36)$$

and the Schrödinger–Robertson uncertainty relation for stretched states reads

$$\sigma_{qq\lambda}\sigma_{pp\lambda} - \sigma_{qp\lambda}^2 = \frac{1}{\lambda^4}\left(\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 + \frac{1}{2}\left(1 - \lambda^2\right)\left(\sigma_{qq} + \sigma_{pp}\right) + \frac{1}{4}\left(1 - \lambda^2\right)^2\right) \geqslant \frac{1}{4\lambda^4}\hbar^2. \quad (37)$$

One can interpret the inequalities (36) and (37) in the sense that the scaling transform $(q; p) \to (\lambda q; \lambda p)$ provides an 'effective Planck's constant' value $\hbar_{\mathrm{eff}} = \hbar/\lambda^2$. For $\lambda^2 \ll 1$ the effective Planck's constant satisfies the inequality

$$\hbar_{\mathrm{eff}} \gg \hbar. \quad (38)$$

A similar situation appeared in the case of correlated states [15]. For these states the value of $\hbar_{\mathrm{eff}} = \hbar/\sqrt{1 - r^2}$ depends on the correlation coefficient $r = \sigma_{xp}/\sqrt{\sigma_x\sigma_p}$ between the coordinate and momentum.

The value of Planck's constant $\hbar$ is responsible for purely quantum phenomena such as quantum tunnelling [16]. The well-known quasiclassical formula for the transmission probability through the potential barrier $U(x)$ reads

$$D \approx \exp\left(-\frac{2}{\hbar}\int_a^b \sqrt{2m(U(x) - E)}\,\mathrm{d}x\right). \quad (39)$$

Here m is the mass of particle and E is its energy.

The above formula together with the inequality (38) shows that for the larger constant $\hbar_{zrmeff}$ the quantum tunnelling effect is enhanced. In [17], it was advocated that the transmission probability for the correlated wave packets with the nonzero correlation coefficient r between the coordinate and the momentum can be higher than for uncorrelated packets, and that the increase of this probability can be described by replacing the true Planck's constant $\hbar$ with the effective constant $\hbar_{\mathrm{eff}} = \hbar/\sqrt{1 - r^2}$. This remark has been done in [15] and developed in [18–22].

We believe that, as correlated states, the stretched states can be used to explain some physical phenomena.

The scaling transformation arises in a natural way in a number of physical problems and especially in the problem of the most quiet phase insensitive amplification of a quantum state [23]. In this case, the parameter λ is equal to the inverse value of the coefficient of amplification $G = 1/\lambda$.

5. Stretched fock states

We will now, as an example, apply the general obtained results to the case of the harmonic oscillator.

It was shown in [24] that a Fock state of the harmonic oscillator is transformed under the scale transformation in the mixed state, which is described by the density matrix

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N + k)!}{k!}\left(1 - \lambda^2\right)^k |N + k\rangle\langle N + k|, \quad \lambda^2 < 1. \quad (40)$$

These Fock stretched states consist of pure states $|N + k\rangle$, $k = 0, 1, 2, \ldots\infty$. Every one of these pure states $|N + k\rangle$ is present in the mixed state with the probability

$$c_k^N = \frac{\lambda^{2N+2}(N + k)!}{N!k!}\left(1 - \lambda^2\right)^k. \quad (41)$$

The distribution of pure states is described by a negative binomial distribution [25]

$$f(k, r, p) = \binom{r + k - 1}{k}p^r q^k; \; p + q = 1; \quad k = 0, 1, 2, \ldots \quad (42)$$

Using the properties of this distribution, it is possible to find the average photon number in a stretched state (40)

$$\begin{aligned}\langle n\rangle &= \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}(N+k)\frac{(N+k)!}{k!}\left(1-\lambda^2\right)^k \\ &= \frac{N+1}{\lambda^2}-1.\end{aligned} \tag{43}$$

And the dispersion of the photon number

$$\sigma_n = \langle n^2\rangle - (\langle n\rangle)^2 = \frac{(N+1)\left(1-\lambda^2\right)}{\lambda^4}. \tag{44}$$

The dispersions σ_{qq}, σ_{pp} and σ_{qp} for the stretched Fock states (40) can be found directly with the help of the expression (40).

$$\langle n|\, q^2\, |n\rangle = n + \frac{1}{2}. \tag{45}$$

$$\begin{aligned}\sigma_{qq\lambda} &= \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\left(N+k+\frac{1}{2}\right)\frac{(N+k)!}{k!}\left(1-\lambda^2\right)^k \\ &= \left(N+\frac{1}{2}\right)+\frac{1}{\lambda^2}(N+1)\left(1-\lambda^2\right) \\ &= \frac{\sigma_{qq}}{\lambda^2}+\frac{1-\lambda^2}{2\lambda^2}.\end{aligned} \tag{46}$$

We see that the expression (46) coincides with the expression (35). The same is true for the dispersions σ_{pp} and σ_{qp}.

6. Conclusion

A new method that allows the Q-symbols of operators to be obtained without resorting to the anti-normal-ordering operation has been developed. In order to achieve this aim, an explicit form of coherent states, in terms of which the Husimi function is used, has been constructed. The proposed formalism is based on using the operators $\hat{X}$ and $\hat{P}$, which constitute a Heisenberg algebra, and are a certain generalization of the standard coordinate and momentum operators. We hope that this approach can also be used to construct and analyze other quasiprobability distributions. With the help of this formalism the mean values of some operators are calculated. It is shown how the uncertainty relations are transformed under scale transformations. The right-hand side of these relations can greatly increase. This fact can be used to study the tunnelling effect. We have also found, in an explicit form, the density matrix for the scaling-transformed Husimi functions of Fock states for a harmonic oscillator. These stretched states can be used in the study of the quantum tunnelling phenomenon.

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