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SCALING TRANSFORM AND STRETCHED STATES IN QUANTUM MECHANICS

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Abstract

We consider the Husimi $Q(q,p)$-functions which are quantum quasiprobability distributions on the phase space. It is known that, under a scaling transform $(q;p) \to (\lambda q;\lambda p)$, the Husimi function of any physical state is converted into a function which is also the Husimi function of some physical state. More precisely, it has been proved that, if $Q(q,p)$ is the Husimi function, the function $\lambda^2 Q(\lambda q;\lambda p)$ is also the Husimi function. We call a state with the Husimi function $\lambda^2 Q(\lambda q;\lambda p)$ the stretched state and investigate the properties of the stretched Fock states. These states can be obtained as a result of applying the scaling transform to the Fock states of the harmonic oscillator. The harmonic-oscillator Fock states are pure states, but the stretched Fock states are mixed states. We find the density matrices of stretched Fock states in an explicit form. Their structure can be described with the help of negative binomial distributions. We present the graphs of distributions of negative binomial coefficients for different stretched Fock states and show the von Neumann entropy of the simplest stretched Fock state.

Keywords: Husimi function, harmonic oscillator, scaling transform, Fock states, stretched states.

1. Introduction

The usual formulation of quantum mechanics is based on the concept of Hilbert space and related structures. In this approach, quantum states are associated with vectors of this space and the observables, with operators in this space. However, there exists the formulation of quantum mechanics similar to the classical statistical mechanics. Quantum mechanics in the statistical theory, and only in it, can predict the probabilities of measurements and, in this context, it is similar to the classical statistical mechanics. The main object of investigation in this theory is the distribution function $\rho(q,p)$. In the classical statistical mechanics, in order to find the mean value of any function $F(q,p)$ defined on the phase space, one has to integrate this function over the phase space weighted with an appropriate probability density function, i.e.,

$$\langle F\rangle = \int F(q,p)\rho(q,p)\,dq\,dp. \tag{1}$$

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In order to describe quantum phenomena in a similar way, one associates a quasidistribution function $D(q,p)$ with each quantum state; the other function $A_D(q,p)$ is called the symbol of the operator $\hat{A}$. We calculate the mean value analogously to (1) and obtain

$$\langle \hat{A} \rangle = \int A_D(q,p) D_\rho(q,p)\, dq\, dp. \tag{2}$$

There exist three well-known quasidistributions: the Wigner function $W(q,p)$ [1], the Husimi–Kano $Q(q,p)$-function [2,3], and the Glauber–Sudarshan $P(q,p)$-function [4,5]. If the operator $\hat{A}$ has the form of the bivariate polynomial of the creation and annihilation operators, its W, Q, and P symbols are obtained using operations of symmetrization and antinormal and normal orderings [6–8].

In this paper, we use the Husimi functions for investigating the properties of stretched Fock states. These states can be obtained as a result of applying the scaling transform $(q;p) \to (\lambda q; \lambda p)$ to usual Fock states of the harmonic oscillator.

2. Husimi Function and the Mean Value Problem

We consider a state described by the density operator $\hat{\rho}$. The Husimi function for this state is determined by the set of coherent states $\langle x|\alpha\rangle$ of the harmonic oscillator [2,3],

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int \langle \alpha | x \rangle \rho(x,y) \langle y | \alpha \rangle\, dx\, dy, \tag{3}$$

where $\alpha = \alpha_r + i\alpha_i$ is an arbitrary complex number, and $\rho(x,y)$ is the kernel of the density operator in the coordinate representation. For the complex number α, which determines the coherent state, we employ the expression $\alpha = (q+ip)/\sqrt{2}$ and consider the Husimi function Q as the function of p and q,

$$Q(q,p) = \frac{1}{2\pi\hbar} \langle q,p|\hat{\rho}|q,p\rangle, \tag{4}$$

where $|q,p\rangle$ is the coherent state given in terms of the variables q and p as follows:

$$\langle x|q,p\rangle = \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}(x-q)^2 + ipx - \frac{i}{2}qp\right], \tag{5}$$

The Husimi function of the state can be described by the wave function $\psi(x)$,

$$Q(q,p) = \frac{1}{2}\left(\frac{1}{\pi}\right)^{3/2} \int \exp\left[-\frac{1}{2}(y-q)^2 - \frac{1}{2}(x-q)^2 + ip(y-x)\right] \psi^*(y)\psi(x)\, dx\, dy. \tag{6}$$

Using the Husimi function, one can evaluate the mean value of an operator by applying formula (2).

In the standard approach, the Q-symbol of the operator $\hat{A}$ can be found, in view of antinormal ordering of the creation and annihilation operators in the expression for the operator $\hat{A}$. The other method of obtaining the Husimi symbols was proposed in [9].

It was proved in [10, 11] that, if $Q(q,p)$ is the Husimi function of the physical state, the value $\lambda^2 Q(\lambda q; \lambda p)$ is also the Husimi function of some physical state. We call the state with the Husimi function $\lambda^2 Q(\lambda q; \lambda p)$ the stretched state.

3. Stretched Fock States

Now as an example, we apply the results obtained to the harmonic oscillator.

In [10, 11], it was shown that the Fock state of the harmonic oscillator is converted under the scaling transform into a mixed state, which is described by the density matrix,

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|, \qquad \lambda^2 < 1. \tag{7}$$

The stretched Fock states contain pure states $|N+k\rangle, \quad k = 0, 1, 2, \ldots, \infty$, and the probability for each pure state $|N+k\rangle$ to enter the stretched Fock states is

$$D_k^N(\lambda) = \frac{\lambda^{2N+2}(N+k)!}{N!k!}(1-\lambda^2)^k. \tag{8}$$

The distribution of pure states is described by the negative binomial distribution [12]:

$$f(k,r,p) = \binom{r+k-1}{k} p^r q^k, \quad p+q = 1, \quad k = 0, 1, 2, \ldots \tag{9}$$

In Fig. 1, we show the distribution of negative binomial coefficients for different values of N and λ.

One can find the mean photon number and its dispersion in the stretched Fock state (7); they read

$$\langle n\rangle = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}(N+k)\frac{(N+k)!}{k!}(1-\lambda^2)^k = \frac{N+1}{\lambda^2} - 1, \tag{10}$$

$$\sigma_n = \langle n^2\rangle - (\langle n\rangle)^2 = \frac{(N+1)(1-\lambda^2)}{\lambda^4}. \tag{11}$$

The dispersions σ_{qq}, σ_{pp}, and σ_{qp} for the stretched Fock states (7) can be found in an explicit form, in view of (7); they are

$$\langle n|q^2|n\rangle = n + 1/2, \tag{12}$$

$$\sigma_{qq\lambda} = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\left(N+k+\frac{1}{2}\right)\frac{(N+k)!}{k!}(1-\lambda^2)^k = \frac{\sigma_{qq}}{\lambda^2} + \frac{1-\lambda^2}{2\lambda^2}. \tag{13}$$

A measure of uncertainty associated with the probability distribution is entropy.

For quantum systems, the von Neumann entropy H is used; in terms of the density matrix $\hat{\rho}$, the von Neumann entropy H reads

$$H = -\sum_k p_k \log_2 p_k, \tag{14}$$

where p_k are the diagonal matrix elements of the density matrix ρ.

It is clear that the von Neumann entropy of the pure Fock state $|N\rangle$ is zero. The entropy of a superposition of the Fock states is defined by the moduli of the coefficients of the states constituting the superposition.

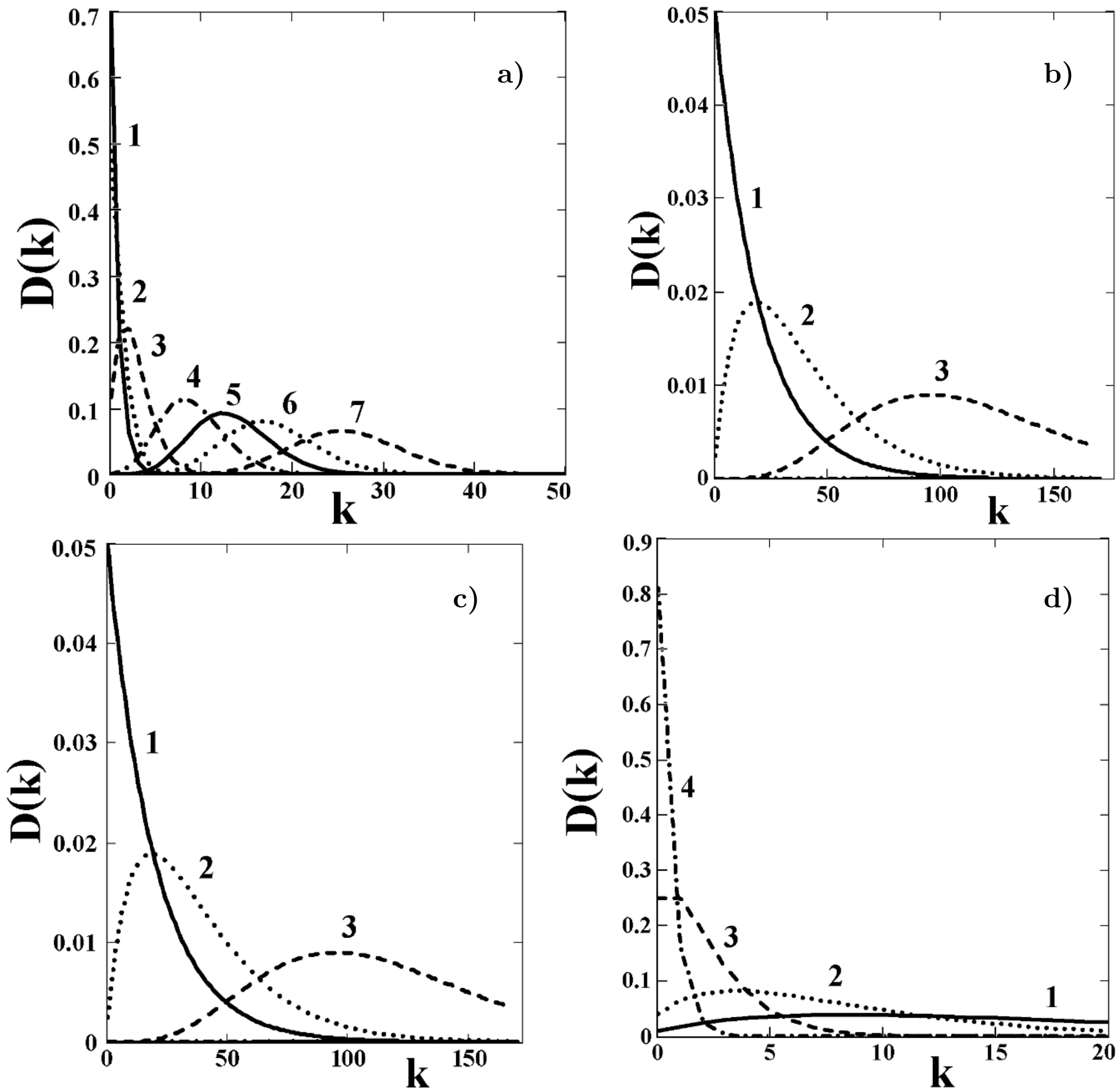

Fig. 1. The negative binomial distribution, where $N = 0$ (curve 1), 1 (curve 2), 5 (curve 3), 20 (curve 4), 30 (curve 5), 40 (curve 6), 60 (curve 7) and $\lambda^2 = 0.1$ (a) and 0.7 (b), $\lambda^2 = 0.05$ and $N = 0$ (curve 1), 1 (curve 2), and 5 (curve 3) (c), and $N = 1$ and $\lambda^2 = 0.1$ (curve 1), 0.2 (curve 2), 0.5 (curve 3), and 0.9 (curve 4) (d).

Now we calculate the von Neumann entropy H of the stretched Fock state and show the result in Fig. 2.

A stretched state is described by the density matrix

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|, \qquad \lambda^2 < 1. \tag{15}$$

Its diagonal matrix elements read

$$D_k^N(\lambda) = \frac{\lambda^{2N+2}(N+k)!}{N!k!}(1-\lambda^2)^k. \tag{16}$$

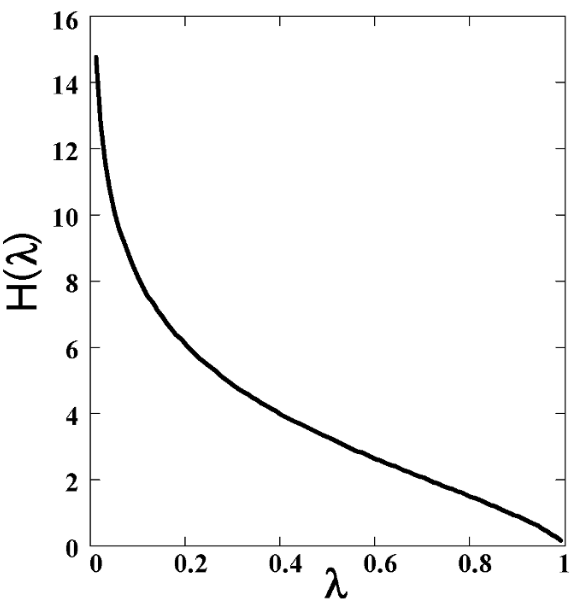

Fig. 2. The von Neumann entropy of the stretched Fock state $\hat{\rho}_0^\lambda$.

Thus, in order to find the von Neumann entropy of the state (15), we need to calculate the value

$$H_N = -\sum_{k=0}^{\infty} D_k^N(\lambda) \log_2 D_k^N(\lambda). \tag{17}$$

If $N = 0$, we have

$$\begin{aligned} H_0 &= -\lambda^2 \sum_{k=0}^{\infty} (1-\lambda^2)^k \log_2 \left(\lambda^2 (1-\lambda^2)^k\right) \\ &= -\log_2 \lambda^2 - \log_2 (1-\lambda^2) \frac{1-\lambda^2}{\lambda^2}. \end{aligned} \tag{18}$$

If $\lambda \to 1$, the value $H_0 \to 0$, and this means that the von Neumann entropy of the N-particle pure state is zero. If $\lambda \to 0$, then $H_0 \to \infty$, and this means that the uncertainty to observe an arbitrary n-particle state increases.

From the viewpoint of information theory, this means that for $\lambda = 1$ we can state that there is only one pure state in the mixed state (15) at $N = 0$, and this state is the vacuum state $|0\rangle$.

If $N > 0$, the von Neumann entropy H_N has a more complicated form and cannot be represented in a closed form. For example, when $N = 1$, the von Neumann entropy H_1 is

$$\begin{aligned} H_1 &= -\lambda^4 \left[\log_2 \lambda^4 \sum_{k=0}^{\infty} (k+1)(1-\lambda^2)^k + \log_2 (1-\lambda^2) \sum_{k=0}^{\infty} k(k+1)(1-\lambda^2)^k \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \log_2 (k+1)(k+1)(1-\lambda^2)^k \right]. \end{aligned} \tag{19}$$

It is obvious that, for any $N > 0$, the behavior of the von Neumann entropy H_N at $\lambda \to 1$ and $\lambda \to 0$ is the same as the behavior of the von Neumann entropy H_0.

4. Conclusions

We investigated the statistical properties of stretched Fock states. These states are constructed as a result of the scaling transform of the Husimi functions of usual Fock states of the harmonic oscillator, and these states are mixed states.

We found the explicit form of the density matrices of the stretched Fock states. The distribution of the pure states in these mixed states is described by the negative binomial distribution.

The stretched Fock states can be employed while studying the quantum tunneling phenomenon [13]. Also the scaling transform arises in the problem of quantum-state amplification [14]; in this case, the parameter λ is equal to the inverse value of the amplification coefficient $G = 1/\lambda$.

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References

1. E. P. Wigner, *Phys. Rev.*, **40**, 749 (1932).
2. K. Husimi, *Proc. Phys. Math. Soc. Jpn.*, **22**, 264 (1940).
3. Y. J. Kano, *J. Math. Phys.*, **6**, 1913 (1965),
4. R. J. Glauber, *Phys. Rev. Lett.*, **10**, 84 (1963).
5. E. C. G. Sudarshan, *Phys. Rev. Lett.*, **10**, 277 (1963).
6. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press (1995), p. 896.
7. M. O. Scully and M. S. Zubairy, *Quantum Optics*, Cambridge University Press (1997), p. 510.
8. W. P. Schleich, *Quantum Optics in Phase Space*, Wiley, Berlin (2001), p. 750.
9. A. V. Andreev, L. D. Davidović, M. D. Davidović, et al., *Theor. Math. Phys.*, **179**, 559 (2014).
10. A. V. Andreev, D. M. Davidović, L. D. Davidović, et al., *Theor. Math. Phys.*, **166**, 356 (2011).
11. A. V. Andreev, D. M. Davidović, L. D. Davidović, and M. D. Davidović, *Phys. Scr.*, **T143**, 014003 (2011).
12. W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York (1957), Vol. 1, p. 480.
13. A. V. Andreev, M. D. Davidović, L. D. Davidović, et al., *Phys. Scr.*, **90**, 074023 (2015).
14. C. G. Agarwal and K. Tara, *Phys. Rev. A*, **47**, 3160 (1993).